

The Betti numbers of $\overline{\mathcal{L}}_n^r$ are real-rooted (hence log-concave)

Draft by Claude (Opus 5), 2026-09-16. Everything below is checked by the scripts in `../compute/`, but has not yet been checked by a human mathematician.

The conjecture

Clader–Damiolini–Li–Ramadas, “Wonderful compactifications and rational curves with cyclic action” (arXiv:2208.05463, *Forum Math. Sigma*), end of §“The Betti numbers of $\overline{\mathcal{L}}_n^r$ ”:

Conjecture. For each r and n , the Betti numbers $\dim A^i(\overline{\mathcal{L}}_n^r)$ form a log-concave sequence.

They note that the $r = 2$ case follows from Eur’s work on the type-B permutohedral variety, where the Betti numbers are the type-B Eulerian numbers. The evidence they give is a table for $r \leq 4$, $n \leq 7$.

Statement

Let $P_{r,n}(t) = \sum_i \dim A^i(\overline{\mathcal{L}}_n^r) t^i$.

Theorem. For every real $r \geq 2$ (in particular every integer $r \geq 2$) and every $n \geq 1$, $P_{r,n}(t)$ has n distinct real roots, all negative. The roots of $P_{r,n-1}$ strictly interlace those of $P_{r,n}$. As a consequence, the Betti numbers of $\overline{\mathcal{L}}_n^r$ are positive and ultra-log-concave, and in particular log-concave. This proves the conjecture.

Proof

Step 1: generating function

CDLR’s additive presentation, from Li Li’s theorem on Chow groups of wonderful compactifications, gives

$$P_{r,n}(t) = \sum_{\ell \geq 0} \sum_{\mathbf{j} \in (\mathbb{Z}_{\geq 2})^\ell} \binom{n}{j_1, \dots, j_\ell, n - |\mathbf{j}|} r^{|\mathbf{j}|} \prod_{i=1}^{\ell} (t + t^2 + \dots + t^{j_i-1}) (1+t)^{n-|\mathbf{j}|}.$$

(We use the corrected multinomial from the arXiv version. `compute/betti.py` reproduces their corrected table exactly.)

This is a binomial convolution of $(1+t)^n$ with a sum over ordered sequences of blocks of size ≥ 2 . By the exponential formula for sequences,

$$F(x, t) := \sum_{n \geq 0} P_{r,n}(t) \frac{x^n}{n!} = \frac{e^{(1+t)x}}{1 - g(rx)}, \quad g(y) = \sum_{j \geq 2} \frac{y^j}{j!} (t + \dots + t^{j-1}).$$

Summing the geometric series in t gives $g(y) = \frac{e^{ty} - te^y + t - 1}{t - 1}$, so

$$F(x, t) = \frac{(t-1)e^{(1+t)x}}{te^{rx} - e^{tx}}.$$

Step 2: a three-term recurrence

A direct computation (`compute/verify_proof.py`, symbolic in r) shows that

$$F_x = (1+t)F + rtx F_x + rt(1-t) F_t + r(r-2)tx F.$$

Taking the coefficient of $x^n/n!$ gives, for $n \geq 0$ (with $P_{r,-1} := 0$),

$$P_{r,n+1}(t) = (1+t+rnt) P_{r,n}(t) + rt(1-t) P'_{r,n}(t) + r(r-2)nt P_{r,n-1}(t).$$

Here $P_{r,0} = 1$, $P_{r,1} = 1+t$, and $P_{r,2} = 1+(r^2+2)t+t^2$. At $r=2$ this is the classical recurrence for type-B Eulerian polynomials.

Step 3: basic properties

- $P_{r,n}(0) = 1$, from the recurrence (or from the formula in Step 1). Positivity of the coefficients is *not* assumed below. It comes out at the end: a monic polynomial whose roots are all negative has positive coefficients.
- $\deg P_{r,n} = n$ with leading coefficient 1. In the recurrence, the t^{n+1} coefficient is $(1+rn) \cdot 1 - rn \cdot 1 = 1$, and the last term has degree only n .

Step 4: interlacing induction

Induction hypothesis for $n \geq 1$: $P_{r,n}$ has roots $x_1 < \dots < x_n < 0$, and $P_{r,n-1}$ has roots $y_1 < \dots < y_{n-1}$ with $x_1 < y_1 < x_2 < \dots < y_{n-1} < x_n$. For $n=1$ the hypothesis holds: $x_1 = -1$ and $P_{r,0} = 1$ has no roots.

Evaluate the recurrence at a root x_i of $P_{r,n}$:

$$P_{r,n+1}(x_i) = \underbrace{r x_i(1-x_i)}_{<0} P'_{r,n}(x_i) + \underbrace{r(r-2) n x_i}_{\leq 0} P_{r,n-1}(x_i).$$

- $P'_{r,n}(x_i) = \prod_{j \neq i} (x_i - x_j)$ has sign $(-1)^{n-i}$, and it is nonzero because the roots are simple. - $P_{r,n-1}(x_i) = \prod_j (x_i - y_j)$ has sign $(-1)^{\#\{j: y_j > x_i\}} = (-1)^{n-i}$ by interlacing.

The first term is strictly nonzero, and both terms have sign $(-1)^{n-i+1}$ (this is where $r \geq 2$ is used). So $\text{sign } P_{r,n+1}(x_i) = (-1)^{n-i+1}$. That gives: - a sign change on each interval (x_i, x_{i+1}) , so $n-1$ roots; - $P_{r,n+1}(x_n) < 0 < 1 = P_{r,n+1}(0)$, so a root in $(x_n, 0)$; - $\text{sign } P_{r,n+1}(x_1) = (-1)^n$, while $P_{r,n+1}(t) \sim t^{n+1}$ has sign $(-1)^{n+1}$ as $t \rightarrow -\infty$, so a root in $(-\infty, x_1)$.

That accounts for all $n+1$ roots. They are real, simple, and negative, and the x_i strictly interlace them. This completes the induction.

Step 5: log-concavity

A polynomial of degree n with positive coefficients $a_0, \dots, a_n$ and only real roots satisfies Newton's inequalities $a_k^2 \geq a_{k-1}a_{k+1}(1 + \frac{1}{k})(1 + \frac{1}{n-k})$. So the sequence is ultra-log-concave, hence log-concave, with no internal zeros. ■

Side results found along the way

- $P_{r,n}(t) = \sum_m \binom{n}{m} r^m d_m(t) (1+t)^{n-m}$, where d_m is the derangement polynomial counting derangements by excedances.
- $P_{r,n}(t) = (1-t) \sum_{k \geq 0} t^k [(rk+1) - (rk+r-1)t]^n$ as formal power series.
- At $r = 1$ the same formulas give the binomial Eulerian polynomials of Postnikov–Reiner–Williams (the h -polynomials of stellohedra). These are also real-rooted by Haglund–Zhang (arXiv:1902.01278), but the proof above does not cover $r = 1$, because the sign of the last recurrence term flips.

Computational evidence (independent of the proof)

- `compute/check.py`: log-concave for all $1 \leq r \leq 15$, $n \leq 40$; real-rooted via Sturm count for $r \leq 15$, $n \leq 25$.
- `compute/interlace_check.py`: the recurrence reproduces the Li Li formula for $2 \leq r \leq 8$, $n \leq 24$, and the roots strictly interlace, computed at 80-digit precision.

Caveats / what to check before showing anyone

1. **Novelty.** Checked 2026-09-16, in more depth after Zack asked:
 - **Citing papers.** Semantic Scholar and OpenAlex list 3 papers citing CDLR (the multimatroid paper, Li–Hahn on tropical Hassett spaces, and the r -braid paper). None resolves the conjecture.
 - **Web searches** on log-concavity and real-rootedness for $\overline{\mathcal{L}}_n^r$ and r -permutohedral fans: nothing.
 - **Shiyue Li’s papers** (Li is a CDLR author): nothing about this conjecture.
 - **Closest danger: Coron–Ferroni–Li**, “Chow polynomials of rank-uniform labeled posets” (arXiv:2511.13819). It proves real-rootedness of (augmented) Chow polynomials of UMEL-shellable posets: uniform matroids, Dowling geometries, etc. It does not mention CDLR.
 - $P_{r,n}$ is **not** the FMV Chow polynomial of the colored-set poset with a top added, nor the augmented Chow polynomial of its rank- n truncation (`compute/fmv_chow.py`, `compute/fmv_chow2.py`). The two agree only at $r = 1$, where both give the stellahedron. At $r = 2$ the truncation gives Athanasiadis’s $\tilde{B}_n^+$ instead.
 - So their theorem doesn’t *obviously* apply, but an expert might see a poset or building-set encoding that I missed.
 - **Coron–Ferroni–Li’s related 2026 papers**, “Matroid analogues of Gal’s conjecture” (arXiv:2604.04550) and “Structural properties of nested set complexes” (arXiv:2601.05188): the first has CDLR in its `.bib` but never cites it in the text; neither mentions the conjecture.

Bottom line: no sign it’s resolved, but this is an active area (Brändén–Vecchi, Ferroni et al.), and those people are exactly who might consider it routine or know an unpublished proof. Worth asking Clader or Li directly.

2. **Dependence.** The proof takes CDLR’s additive Chow-group formula (their application of Li Li’s theorem) as given.

Part II: all π -colored fans (multimatroid setting)

Added the same day, 2026-09-16.

Clader–Damiolini–Eur–Huang–Li, “Multimatroids and rational curves with cyclic action” (arXiv:2311.09314), define a fan Σ^π for any partition $E = E_1 \sqcup \cdots \sqcup E_n$. Its cones are chains of *colored* sets, meaning sets with at most one element from each E_i . The uniform case $|E_i| = r$ is the fan Σ_n^r with $A^*(\Sigma_n^r) \cong A^*(\overline{\mathcal{L}}_n^r)$. Write $r_i = |E_i|$ and $P_\pi(t) = \sum_k \dim A^k(\Sigma^\pi) t^k$.

Theorem 2. If every $r_i \geq 2$ (real $r_i \geq 2$ suffices for the polynomial statement), then P_π has only simple negative real roots. Its coefficients are therefore ultra-log-concave, and $P_{\pi \setminus i}$ interlaces P_π for every i .

Step 1: Betti numbers

Consider the product arrangement in $(\mathbb{P}^1)^n$ that has r_i distinct points on the i -th factor. Its flats are $\{x_i = \alpha(i), i \in I\}$ for $I \subseteq [n]$ and $\alpha(i) \in (\text{points on factor } i)$, i.e. exactly the nonempty colored sets. Take the maximal building set: nested sets are then chains, so the nested set fan is Σ^π . By CDLR’s product-arrangement theorem (labeled `thm:wcproduct` in the arXiv source, stated for arbitrary essential arrangements $\mathcal{A}_i \subset \mathbb{P}^{k_i}$), $A^*(X_{\Sigma^\pi}) \cong A^*(\overline{Y}_\pi)$. Li Li’s theorem applies exactly as in CDLR’s uniform computation, with $r^{|I|}$ replaced by $\prod_{i \in I} r_i$. This gives

$$P_\pi(t) = \sum_{A \subseteq [n]} \left(\prod_{i \in A} r_i \right) d_{|A|}(t) (1+t)^{n-|A|}.$$

Check: `compute/nonuniform.py` computes $A^*(\Sigma^\pi)$ directly (Stanley–Reisner ideal plus linear forms, rank mod p) for $\pi = (2, 2), (3, 3), (3, 2), (4, 2), (2, 2, 2), (3, 2, 2)$, and every case matches.

Step 2: product formula

Binomial inversion ($d_m = \sum_j \binom{m}{j} (-1)^{m-j} A_j$) together with $A_j(t) = (1-t)^{j+1} \sum_k (k+1)^j t^k$, applied one class at a time, gives

$$\frac{P_\pi(t)}{1-t} = \sum_{k \geq 0} t^k \prod_{i=1}^n L_k(r_i), \quad L_k(r) = (rk+1) - (rk+r-1)t.$$

(`compute/product_formula_check.py`, symbolic in the r_i .)

Step 3: recurrence for adding a color class

Write $S_\pi = \sum_k t^k \prod_i L_k(r_i)$, $\Pi_k = \prod_i L_k(r_i)$, and $D = t \frac{d}{dt}$. - $L_k(r) = (1-t)(rk+1) - (r-2)t$ gives $S_{\pi+s} = s(1-t) \sum_k k t^k \Pi_k + (1-(s-1)t) S_\pi$. - $\partial_t L_k(r_i) = -(r_i k + r_i - 1) = -\frac{L_k(r_i) + r_i - 2}{1-t}$ gives $DS_\pi = \sum_k k t^k \Pi_k - \frac{t}{1-t} [n S_\pi + \sum_i (r_i - 2) S_{\pi \setminus i}]$.

Eliminate $\sum_k k t^k \Pi_k$ and substitute $S = P/(1-t)$:

$$P_{\pi+s} = (1+t+snt) P_\pi + st(1-t) P'_\pi + st \sum_{i=1}^n (r_i - 2) P_{\pi \setminus i}.$$

In the uniform case the last sum is $n(r-2)P_{r,n-1}$, recovering Part I. (`compute/nonuniform_proof_check.py` checks this identity symbolically for $n \leq 4$.)

Step 4: induction on n

Claim $C(n)$: for every multiset π of size n with all $r_i \geq 2$, P_π has n simple negative roots, and $P_{\pi \setminus i}$ strictly interlaces P_π for every i .

$C(1)$ holds, since $P_{(r)} = 1 + t$. Now let π have size $n + 1$ and fix any class j . Write $\pi = \rho + r_j$ with $|\rho| = n$. At each root x of P_ρ , the recurrence gives

$$P_\pi(x) = r_j x(1 - x) P'_\rho(x) + r_j x \sum_i (r_i - 2) P_{\rho \setminus i}(x).$$

By $C(n)$, every $P_{\rho \setminus i}(x)$ has the same sign as $P'_\rho(x)$. Since $x < 0$ and $r_i \geq 2$, both terms share a sign and the first is nonzero. The sign-counting argument from Part I (with $\deg = n + 1$, leading coefficient 1, and $P_\pi(0) = 1$) then shows that P_π is real-rooted with simple negative roots, strictly interlaced by $P_\rho = P_{\pi \setminus j}$. Since j was arbitrary and P_π is symmetric in the r_i , this proves $C(n + 1)$. ■

Numerics. Real roots in 1,500 random nonuniform π with $n \leq 14$ and $r_i \leq 20$. With some $r_i = 1$ mixed in, the formula still gives real-rooted polynomials numerically, but that case is degenerate for Σ^π and not covered by the proof.

Part III: corollaries and parked leads

- **γ -positivity, with explicit coefficients.** d_m is γ -positive with center $m/2$: $d_m = \sum_i \xi_{m,i} t^i (1 + t)^{m-2i}$, where $\xi_{m,i} \geq 0$ counts derangements of $[m]$ with i excedances and no double excedances $j < w(j) < w(w(j))$ (see Athanasiadis's γ -positivity survey; checked by brute force for $m \leq 8$). Multiplying by $(1 + t)^{n-m}$ preserves center $n/2$, so

$$P_\pi = \sum_i \left(\sum_{A \subseteq [n]} (\prod_{i' \in A} r_{i'}) \xi_{|A|,i} \right) t^i (1 + t)^{n-2i}.$$

This follows immediately once the derangement formula is in hand.

- **Parked conjecture: real roots for $\overline{\mathcal{M}}_n^r$.** Here $\overline{\mathcal{M}}_n^r$ is the wonderful compactification of the r -braid (Dowling) arrangement with its minimal building set (arXiv:2506.18205). From the Feichtner–Yuzvinsky basis I derived

$$\sum_n P(\overline{\mathcal{M}}_n^r) \frac{x^n}{n!} = \frac{1 - t}{e^{(t-1)T} - t}, \quad x = \frac{(e^{rtT} - 1) - t^2(e^{rT} - 1)}{rt(1 - t)}.$$

This matches brute-force nested-set enumeration (`compute/wonderful.py`, `compute/mrn_series.py`), and at $r = 1$ it is Manin's formula for $\overline{\mathcal{M}}_{0,n+2}$. The polynomials are real-rooted for $r \in \{2, 3, 4\}$, $n \leq 13$ (`compute/mrn_rr.py`). The $r = 1$ case is the Aluffi–Chen–Marcolli conjecture, proved only in 2026 with substantial effort (Bérczi–Kiem, arXiv:2605.29151), so this is probably not cheap.

- **Equivariant version** (e.g. $S(r, n)$ acting on cohomology): not attempted; it would take a lot of study.