

The Betti Numbers $\dim A^i(\overline{\mathcal{L}}_n^r)$ Form a Log-Concave Sequence

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Inspired by the exploits of Levent Alpöge (an employee of the artificial intelligence company Anthropic who has recently published several high-profile AI-assisted breakthroughs, including a counterexample to the Jacobian conjecture¹ and the existence² of a complex structure on S^6), ZMD was curious whether publicly available language models could make progress on open problems, and prompted Claude Opus 5 to look for candidates in the recent work of Emily Clader. Within less than ten minutes, the model claimed to have a proof of Conjecture 4.15 in Clader *et al.* “Wonderful Compactifications and Rational Curves with Cyclic Action.” This writeup represents ZMD’s attempt to make sense of the model’s work (with extensive tutoring from other Claude instances).

Theorem 4.13 states that for any $k \in \mathbb{N}_0$, there exists a group isomorphism

$$A^k(\overline{\mathcal{L}}_n^r) \cong A^k\left((\mathbb{P}^1)^n\right) \oplus \bigoplus_{\ell \geq 1} \bigoplus_{\mathbf{j} \in (\mathbb{N}_{\geq 2})^\ell} \left(\bigoplus_{\boldsymbol{\mu} \in M_{\mathbf{j}}} A^{k-|\boldsymbol{\mu}|} \left((\mathbb{P}^1)^{n-|\mathbf{j}|} \right) \right)^{\oplus N_{\mathbf{j}}} \quad (1)$$

where the absolute value bars denote the sum of the entries of a tuple, $M_{\mathbf{j}} := \{\boldsymbol{\mu} \in \mathbb{Z}^\ell : \forall i \in \{1 \dots \ell\} 1 \leq \mu_i < j_i\}$, and $N_{\mathbf{j}} := \binom{n}{j_1, \dots, j_\ell, n-|\mathbf{j}|} r^{|\mathbf{j}|}$.

To be clear, I don’t really know much about the algebraic geometry here, but I’m given to understand that these are free abelian groups: if they’re isomorphic, they have the same rank (and $\dim(G \oplus H) = \dim G + \dim H$):

$$\dim A^k(\overline{\mathcal{L}}_n^r) = \dim A^k\left((\mathbb{P}^1)^n\right) + \sum_{\ell \geq 1} \sum_{\substack{\mathbf{j} \in (\mathbb{N}_{\geq 2})^\ell \\ |\mathbf{j}| \leq n}} N_{\mathbf{j}} \sum_{\boldsymbol{\mu} \in M_{\mathbf{j}}} \dim A^{k-|\boldsymbol{\mu}|} \left((\mathbb{P}^1)^{n-|\mathbf{j}|} \right)$$

Then we can form the Poincaré polynomial $P_{r,n}(t)$: a generating function in which (for a given r and n), the coefficient of t^k is $\dim A^k(\overline{\mathcal{L}}_n^r)$.

$$P_{r,n}(t) := \sum_k \dim A^k(\overline{\mathcal{L}}_n^r) t^k = \sum_k \dim A^k\left((\mathbb{P}^1)^n\right) t^k + \sum_{\ell \geq 1} \sum_{\substack{\mathbf{j} \in (\mathbb{N}_{\geq 2})^\ell \\ |\mathbf{j}| \leq n}} N_{\mathbf{j}} \sum_{\boldsymbol{\mu} \in M_{\mathbf{j}}} \sum_k \dim A^{k-|\boldsymbol{\mu}|} \left((\mathbb{P}^1)^{n-|\mathbf{j}|} \right) t^k$$

Because $\mathbb{P}^1 \cong S^2$, its Poincaré polynomial is $1 + s^2$, but since all the homology is in even degree, we use $t := s^2$ and get $1 + t$. The polynomial of the product of spaces is the product of the polynomials (Künneth), so $\sum_k \dim A^k\left((\mathbb{P}^1)^p\right) t^k = (1 + t)^p$, which we can apply to our first summation (with $p = n$). The same insight (with $p = n - |\mathbf{j}|$) can be applied to the later summation, but we have to reindex it: if we say that $k' := k - |\boldsymbol{\mu}|$, then $k = k' + |\boldsymbol{\mu}|$. Thus $t^k = t^{k'+|\boldsymbol{\mu}|} = t^{k'} t^{|\boldsymbol{\mu}|}$. Then $\sum_k \dim A^{k-|\boldsymbol{\mu}|} \left((\mathbb{P}^1)^{n-|\mathbf{j}|} \right) t^k = t^{|\boldsymbol{\mu}|} \sum_{k'} \dim A^{k'} \left((\mathbb{P}^1)^{n-|\mathbf{j}|} \right) t^{k'} = t^{|\boldsymbol{\mu}|} (1 + t)^{n-|\mathbf{j}|}$. So we have

$$(1 + t)^n + \sum_{\ell \geq 1} \sum_{\substack{\mathbf{j} \in (\mathbb{N}_{\geq 2})^\ell \\ |\mathbf{j}| \leq n}} \binom{n}{j_1, \dots, j_\ell, n-|\mathbf{j}|} r^{|\mathbf{j}|} (1 + t)^{n-|\mathbf{j}|} \sum_{\boldsymbol{\mu} \in M_{\mathbf{j}}} t^{|\boldsymbol{\mu}|}$$

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¹https://x.com/_alpoge_/status/2079028340955197566

²https://x.com/_alpoge_/status/2091639597193368014

Let's interrogate $\sum_{\mu \in M_j} t^{|\mu|}$. Every μ is an ℓ -tuple of integers, where the i th entry is between 1 (inclusive) and j_i (exclusive, for the j from an outer summation). We have $\sum_{\mu \in M_j} t^{|\mu|} = \sum_{\mu \in M_j} t^{\sum_{i=1}^{\ell} \mu_i} = \sum_{\mu \in M_j} \prod_{i=1}^{\ell} t^{\mu_i}$, and then interchanging the sum and product by the distributive law, $\sum_{\mu \in M_j} \prod_{i=1}^{\ell} t^{\mu_i} = \prod_{i=1}^{\ell} \sum_{m=1}^{j_i-1} t^m$. So we have

$$(1+t)^n + \sum_{\ell \geq 1} \sum_{\substack{j \in (\mathbb{N}_{\geq 2})^{\ell} \\ |j| \leq n}} \binom{n}{j_1, \dots, j_{\ell}, n-|j|} r^{|j|} (1+t)^{n-|j|} \prod_{i=1}^{\ell} \sum_{m=1}^{j_i-1} t^m$$

For $\ell := 0$, $\binom{n}{j_1, \dots, j_{\ell}, n-|j|} r^{|j|} (1+t)^{n-|j|} \prod_{i=1}^{\ell} \sum_{m=1}^{j_i-1} t^m = \binom{n}{n-0} r^0 (1+t)^{n-0} = (1+t)^n$, so the leading $(1+t)^n$ term can be absorbed into the outer summation:

$$P_{r,n}(t) = \sum_{\ell \geq 0} \sum_{\substack{j \in (\mathbb{N}_{\geq 2})^{\ell} \\ |j| \leq n}} \binom{n}{j_1, \dots, j_{\ell}, n-|j|} r^{|j|} (1+t)^{n-|j|} \prod_{i=1}^{\ell} \sum_{m=1}^{j_i-1} t^m$$

Also, $r^{|j|} = r^{\sum_{i=1}^{\ell} j_i} = \prod_{i=1}^{\ell} r^{j_i}$, which can be merged into the product, and we can write out the multinomial coefficient in terms of factorials:

$$P_{r,n}(t) = \sum_{\ell \geq 0} \sum_{\substack{j \in (\mathbb{N}_{\geq 2})^{\ell} \\ |j| \leq n}} \frac{n!}{(n-|j|)! \prod_{i=1}^{\ell} j_i!} (1+t)^{n-|j|} \prod_{i=1}^{\ell} r^{j_i} \sum_{m=1}^{j_i-1} t^m \quad (2)$$

At this point, we've constructed a generating function $P_{r,n}(t)$ that reproduces individual rows of the table presented in Example 4.14 (where t picks out the t th entry in the row).

Next, we'll construct an exponential generating function $F(x, t)$ that reproduces the entire table (where x will pick out the x th row).

We begin by forming the exponential generating function by multiplying by $\frac{x^n}{n!}$ and summing over n :

$$F(x, t) := \sum_{n \geq 0} \sum_{\ell \geq 0} \sum_{\substack{j \in (\mathbb{N}_{\geq 2})^{\ell} \\ |j| \leq n}} \frac{n!}{(n-|j|)! \prod_{i=1}^{\ell} j_i!} \frac{x^n}{n!} (1+t)^{n-|j|} \prod_{i=1}^{\ell} r^{j_i} \sum_{m=1}^{j_i-1} t^m$$

But note that $n = n - |j| + \sum_{i=1}^{\ell} j_i$, and therefore $x^n = x^{n-|j|} \prod_{i=1}^{\ell} x^{j_i}$:

$$F(x, t) = \sum_{n \geq 0} \sum_{\ell \geq 0} \sum_{\substack{j \in (\mathbb{N}_{\geq 2})^{\ell} \\ |j| \leq n}} (1+t)^{n-|j|} \frac{x^{n-|j|}}{(n-|j|)!} \prod_{i=1}^{\ell} \frac{x^{j_i}}{j_i!} r^{j_i} \sum_{m=1}^{j_i-1} t^m$$

Then we do some more reindexing: let $p := n - |j|$. Then we can pull out the subexpressions now newly in terms of p :

$$F(x, t) = \left(\sum_{p \geq 0} (1+t)^p \frac{x^p}{p!} \right) \sum_{\ell \geq 0} \sum_{j \in (\mathbb{N}_{\geq 2})^{\ell}} \prod_{i=1}^{\ell} \frac{x^{j_i}}{j_i!} r^{j_i} \sum_{m=1}^{j_i-1} t^m$$

But by the power series formulation of the natural exponential, $\sum_{p \geq 0} (1+t)^p \frac{x^p}{p!} = e^{(1+t)x}$:

$$F(x, t) = e^{(1+t)x} \sum_{\ell \geq 0} \sum_{j \in (\mathbb{N}_{\geq 2})^{\ell}} \prod_{i=1}^{\ell} \frac{x^{j_i}}{j_i!} r^{j_i} \sum_{m=1}^{j_i-1} t^m$$

Interchanging the sum and product by the distributive law:

$$= e^{(1+t)x} \sum_{\ell \geq 0} \prod_{i=1}^{\ell} \sum_{j \geq 2} \frac{x^j}{j!} r^j \sum_{m=1}^{j-1} t^m$$

We have a geometric series:

$$e^{(1+t)x} \sum_{\ell \geq 0} \left(\sum_{j \geq 2} \frac{x^j}{j!} r^j \sum_{m=1}^{j-1} t^m \right)^\ell = e^{(1+t)x} \cdot \frac{1}{1 - \sum_{j \geq 2} \frac{x^j}{j!} r^j \sum_{m=1}^{j-1} t^m}$$

Which we can further simplify ($\sum_{m=1}^{j-1} t^m$ is a finite geometric series equal to $t \frac{1-t^{j-1}}{1-t} = \frac{t-t^j}{1-t}$):

$$\begin{aligned} \frac{e^{(1+t)x}}{1 - \sum_{j \geq 2} \frac{(rx)^j}{j!} \frac{t-t^j}{1-t}} &= \frac{e^{(1+t)x}}{1 - \sum_{j \geq 2} \frac{t(rx)^j - (trx)^j}{(1-t)j!}} = \frac{e^{(1+t)x}}{1 - \frac{1}{1-t} \left(t \sum_{j \geq 2} \frac{(rx)^j}{j!} - \sum_{j \geq 2} \frac{(trx)^j}{j!} \right)} \\ &= \frac{e^{(1+t)x}}{1 - \frac{1}{1-t} (t(e^{rx} - 1 - rx) - (e^{trx} - 1 - trx))} = \frac{e^{(1+t)x}}{1 - \frac{1}{1-t} (te^{rx} - t - te^{trx} + 1 + trx)} \\ &= \frac{e^{(1+t)x}}{\underbrace{\frac{1-t}{1-t} - \frac{1}{1-t} (te^{rx} - t - e^{trx} + 1)}_{\text{"fancy one"}}} = \frac{e^{(1+t)x}}{\frac{1-t - te^{rx} + t + e^{trx} - 1}{1-t}} = \frac{(1-t)e^{(1+t)x}}{e^{trx} - te^{rx}} \end{aligned}$$

So we've constructed this exponential generating function

$$F(x, t) = \frac{(1-t)e^{(1+t)x}}{e^{trx} - te^{rx}} \quad (3)$$

Next, we're going to use a partial differential equation involving F to construct a recurrence relation for P .

Let $D := e^{rtx} - te^{rx}$ (D for the denominator of F). Its partial derivatives are $D_x = rte^{rtx} - rte^{rx} = rt(e^{rtx} - e^{rx})$ and $D_t = rxe^{rtx} - e^{rx}$.

Now all three of D , D_x , and D_t are combinations of e^{rtx} and e^{rx} , so are linearly dependent (in the sense of a vector space with coefficients of rational functions of x and t), so there should be a relation among them: $\alpha D_x + \beta D_t = \gamma D$ for some α , β , and γ not all zero. Matching the coefficients of each exponential, we have $\begin{cases} \alpha rt + \beta rx = \gamma \\ -\alpha rt - \beta = -t\gamma \end{cases}$. Negating the second equation and substituting the first into the second, we have $\alpha rt + \beta = \alpha rt^2 + \beta trx$, which is $\alpha rt(1-t) = \beta(trx - 1)$. We can choose $\alpha := rtx - 1$ to get $\beta = rt(1-t)$. Then $\gamma = (rtx - 1)rt + rt(1-t)rx = r^2t^2x - rt + r^2tx - r^2t^2x$, so we have

$$(rtx - 1)D_x + rt(1-t)D_t = rt(rx - 1)D \quad (4)$$

We know that

$$FD = (1-t)e^{(1+t)x} \quad (5)$$

Differentiating (5) with respect to x , we have $F_x D + F D_x = (1+t) \underbrace{(1-t)e^{(1+t)x}}_{FD} = (1+t)FD$. Then dividing

by FD gives $\frac{F_x}{F} + \frac{D_x}{D} = 1+t$, which rearranges to $\frac{D_x}{D} = 1+t - \frac{F_x}{F}$.

But differentiating (5) with respect to t , we also have $F_t D + F D_t = -e^{(1+t)x} + (1-t)xe^{(1+t)x} = -\frac{FD}{1-t} + (1-t)x\frac{FD}{1-t} = -\frac{FD}{1-t} + xFD = FD\left(-\frac{1}{1-t} + x\right)$. Then dividing by FD gives us $\frac{F_t}{F} + \frac{D_t}{D} = -\frac{1}{1-t} + x$, which rearranges to $\frac{D_t}{D} = -\frac{1}{1-t} + x - \frac{F_t}{F}$.

Dividing (4) by D yields $(rtx - 1)\frac{D_x}{D} + rt(1-t)\frac{D_t}{D} = rt(rx - 1)$ and substituting yields $(rtx - 1)\left(1+t - \frac{F_x}{F}\right) + rt(1-t)\left(-\frac{1}{1-t} + x - \frac{F_t}{F}\right) = rt(rx - 1)$.

We do some more algebra:

$$rtx + rt^2x - rtx\frac{F_x}{F} - 1 - t + \frac{F_x}{F} + rt + rt(1-t)x - rt(1-t)\frac{F_t}{F} = r^2tx$$

$$-rtx\frac{F_x}{F} + \frac{F_x}{F} - rt(1-t)\frac{F_t}{F} = r^2tx - rt(1-t)x + 1 + t - rtx - rt^2x = r^2tx - rtx + rt^2x + 1 + t - rtx - rt^2x = r(r-2)tx + 1 + t$$

Then multiplying by F ,

$$(1 - rt)x F_x - rt(1 - t)F_t = r(r - 2)txF + F + tF$$

And rearranging,

$$F_x = r(r - 2)txF + (1 + t)F + rtxF_x + rt(1 - t)F_t \quad (6)$$

This partial differential equation for F corresponds to a recurrence relation for P . For instance, $F_x = \sum_{n \geq 1} P_{r,n}(t) \frac{x^{n-1}}{(n-1)!}$, which can be reindexed to $\sum_{n \geq 0} P_{r,n+1}(t) \frac{x^n}{n!}$, so the coefficient on $\frac{x^n}{n!}$ in F_x corresponds to $P_{r,n+1}$. Similar reasoning leads us to the recurrence

$$P_{r,n+1}(t) = (1 + t + rnt)P_{r,n}(t) + rt(1 - t)P'_{r,n}(t) + r(r - 2)ntP_{r,n-1}(t)$$

Finally, we're ready for

Lemma. $P_{r,n}$ has n distinct real roots, all negative.

Proof. By induction that $P_{r,n}$ and $P_{r,n+1}$ each have only negative real roots, which “interlace”: if the roots of $P_{r,n}$ are $(a_i)_{i=1}^n$ and the roots of $P_{r,n+1}$ are $(b_i)_{i=1}^{n+1}$, then $b_1 < a_1 < b_2 < \dots < b_n < a_n < b_{n+1}$.

(Base.) $P_{r,0} = F(0, t) = \frac{(1-t)e^{(1+t)0}}{e^{rt0} - te^{r0}} = \frac{1-t}{1-t} = 1$ has no roots, which vacuously interlace the one root of

$$P_{r,1} = (1 + t + 0rt)P_{r,0}(t) + \cancel{rt(1-t)P'_{r,0}(t)} + 0r(r-2)tP_{r,-1}(t) = 1 + t, \text{ which is } -1.$$

(Induction.) Suppose that $P_{r,n}$ (roots $(a_i)_{i=1}^n$) and $P_{r,n+1}$ (roots $(b_i)_{i=1}^{n+1}$) have negative interlacing real roots.

We look at $P_{r,n+2}$ evaluated at b_i : $P_{r,n+2}(b_i) = \cancel{(1 + b_i + r(n+1)b_i)P_{r,n+1}(b_i)} + rb_i(1 - b_i)P'_{r,n+1}(b_i) + r(r - 2)(n + 1)b_iP_{r,n}(b_i)$. We know that $rb_i(1 - b_i) < 0$: positive (r) times negative (b_i) times positive ($1 - b_i$). We know that $r(r - 2)(n + 1)b_i \leq 0$. (If $r = 2$, then $r - 2$ is zero. Otherwise, that's positive times positive times positive times negative.)

$P_{r,n+1}$ has a leading coefficient of 1, so it factors as $\prod_{i=1}^{n+1} (t - b_i)$. Differentiating with the product rule produces $n + 1$ terms, but all but one vanish when we evaluate at $t := b_i$: $P'_{r,n+1}(b_i) = \prod_{j \neq i} (b_i - b_j)$. The factor $b_i - b_j$ is negative when $j > i$, which happens $(n + 1) - i$ times, so $\text{sign } P'_{r,n+1}(b_i) = (-1)^{n+1-i}$.

Then $P_{r,n}(t) = \prod_{j=1}^n (t - a_j)$, and we have a similar argument: evaluating at b_i gives $\prod_{j=1}^n b_i - a_j$. The number of negatives in the product is the number of a_j s greater than b_i —which is $n - (i - 1) = n + 1 - i$. (The a_j s go up to n , but the ones up to a_{i-1} are less than b_i such that $b_i - a_j$ is positive.) So $\text{sign } P_{r,n}(b_i) = (-1)^{n+1-i}$ as well.

Putting the signs together, we see that $\text{sign } P_{r,n+2}(b_i) = (-1)^{n+2-i}$. (Either both terms have that sign, or the $P'_{r,n+1}(b_i)$ term does and the other is zero.) By the intermediate value theorem, each of the intervals $(-\infty, b_1)$, $(b_1, b_2) \dots (b_n, b_{n+1})$, $(b_{n+1}, 0)$ contains exactly one root of $P_{r,n+2}$. which we'll call $(c_k)_{k=1}^{n+2}$, which are the $n + 2$ roots of $P_{r,n+2}$. So we have interlacing $c_1 < b_1 < c_2 \dots < c_{n+1} < b_{n+1} < c_{n+2}$.

But if $P_{r,n}$ has only negative real roots $(-d_i)_{i=1}^n$ and leading coefficient 1, then it's $\prod_{i=1}^n (t + d_i)$, and has positive coefficients. A well-known result of Newton's inequalities is that if the polynomial $\sum_{k=0}^n a_k t^k$ has only real roots, then $a_k^2 \geq a_{k-1}a_{k+1} \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) > a_{k-1}a_{k+1}$, and therefore $2 \log a_k \geq \log(a_{k-1}a_{k+1})$ and therefore $\log a_k \geq \frac{\log a_{k-1} + \log a_{k+1}}{2}$, which is to say that the Betti numbers of $A^i(\mathcal{L}_n^r)$ form a log-concave sequence, which is *quod erat demonstrandum*.

Appendix: Initial Claude Code Prompt

The following file was created as intro.md. A Claude Code session was started with the initial message “Read intro.md. Does the idea make sense? Like, do you see why I would be asking Claude this?”

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## Introduction to Project Vibescop
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Recently, Anthropic employee Levent Alpöge has shocked the math world by Tweeting out the solutions to multiple significant open problems found in collaboration with internal Claude models (search for context). As a scholar-aristocrat of mediocre math talent who hangs around the SFSU math department (read Diary_04B.md for context), I wonder if there's a lot more low-hanging fruit that could be plucked by publicly-available LLMs that just hasn't been plucked yet, because there are sufficiently few AI-pilled math nerds that no one bothered spending the tokens. (My impression from hanging out at SFSU is that everyone knows about LLMs as homework help—that's why they can't do take-home exams anymore—but few people are
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thinking about the big picture.)

The mission in this repo is to, (1) investigate the recent work of Prof. Emily Clader and try to find open problems that might be interesting to her that might be scoopable with an LLM (so, like, stuff where you can compute a checkable counterexample, which would have been a master's student project), and (2) actually scoop the result (which will probably demand expensive Fable tokens, although Opus is probably fine for part (1)).

My role as a human is to be mathematically literate enough to be able to read what Claude comes up with given a few hours of background study even if it would take me months or years to do the work myself (see `algebraic_topology-01.lyx` for calibration of how skilled I am). I'm just really curious if it's possible to find anything good enough to be worth taking to Prof. Clader's office hours.

A previous Claude suggested "Multimatroids and rational curves with cyclic action" (arXiv:2311.09314, IMRN 2024) as a good paper to start with, but you should look for more Clader papers.

If the authors say things like, "It would be interesting to investigate ...", that's the kind of thing we're looking for (because that's stuff that would presumably be farmed out to a human Master's student).