Exercises in Measure Theory

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Abstract

Exercises from Bernd S. W. Schröder, *Mathematical Analysis: A Concise Introduction*

14-4. **a**. Given a measure space (M, Σ, μ) with $\Omega \in \Sigma$, we want to show that $\Sigma^\Omega := \{S \in \Sigma : S \subseteq \Omega\}$ is a sigma-algebra.

(Empty set.) $\emptyset \in \Sigma^\Omega$ because $\emptyset \in \Sigma$ and $\emptyset \subseteq \Omega$ vacuously.

I assume that in this context, $S^\complement$ is supposed to be $\Omega \backslash S$, not $M \backslash S$; that's the only intepretation that makes sense.

(Closure under complementation.) We want to show that $\Omega \backslash S \in \Sigma$ and $\Omega \backslash S \subseteq \Omega$. The latter is trivial.

For the former, we know that $S \in \Sigma^\Omega$ is also in Σ. So $M \backslash S$ is also in Σ. Sigma-algebras are closed under intersections, so $M \backslash S \cap \Omega = \Omega \backslash S$ is also in Σ.

(Countable union.) $\forall n\ A_n \in \Sigma$ implies $\cup_n A_n \in \Sigma$ (by countable additivity of Σ) and $\forall n\ A_n \subseteq \Omega$ furthermore implies $\cup_n A_n \in \Sigma^\Omega$.

b. $\mu_\Omega := \mu|_{\Sigma^\Omega}$ is a measure.

(Empty set.) $\mu(\emptyset) = 0$ implies that $\mu|_{\Sigma^\Omega}(\emptyset) = 0$ (restricting the domain doesn't change the value of the function for $\emptyset$).

(Countable additivity.) If for all n, $A_n \in \Sigma^\Omega$, then $\mu_\Omega(\cup_n A_n) = \sum \mu(A_n)$ (restricting the domain doesn't change the value of the function for inputs in the new domain).

14-6. We want to show that $\mu(\cup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu(A_n)$. If the A_n are disjoint, equality holds by countable additivity. So suppose the A_n are not disjoint. Then the "idea" of the proof is obviously going to be that the inequality holds because $\sum_{i=1}^\infty \mu(A_i)$ double- or triple-*&c.* counts the overlaps, but I've been having a lot of trouble formalizing this!

Let $A_{\neg i} := \cup_{j \neq i} A_j$. For all $i \in \mathbb{N}$, $A_i = A_i \backslash A_{\neg i} \sqcup (A_i \cap A_{\neg i})$. So $\sum_{i=1}^\infty \mu(A_i) = \sum_i \mu(A_i \backslash A_{\neg i}) + \mu(A_i \cap A_{\neg i})$. By the inclusion–exclusion principle, $\mu(\bigcup_{n=1}^\infty A_n) = \mu(\sum_{i=1}^\infty (-1)^{i+1} \left(\sum_{j_1 < \ldots < j_i} |\bigcap A_{j_k}|_{k=1}^i \right)) \ldots$

At this point, I asked for a hint from Anthropic Claude 3.5 Sonnet, and it claimed that $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty A_i \backslash A_{\neg i}$, which wasn't clear to me before ... because it's false. (What if all the sets are the same?) Then I asked OpenAI o1, and it suggested a slightly different way to "disjointify" the union: only exclude the intersection of "earlier" sets, not all other sets. So we have $\mu(\bigcup_{i=1}^\infty A_i) = \mu(\bigsqcup_{i=1}^\infty A_i \backslash \bigcup_{j<i} A_j)$. By countable additivity,

$$\mu(\bigsqcup_{i=1}^\infty A_i \backslash \bigcup_{j<i} A_j) = \sum_{i=1}^\infty \mu(A_i \backslash \bigcup_{j<i} A_j) \leq= \sum_{i=1}^\infty \mu(A_i)$$

14-7. We want to show that $\Sigma \subseteq \mathcal{P}(M)$ is a σ-algebra iff $\emptyset \in \Sigma$; if $S \in \Sigma$ then $S^\complement \in \Sigma$; and $\forall n\ A_n \in \Sigma \Rightarrow \bigcap_n A_n \in \Sigma$.

($\Rightarrow$) The first two properties are in the σ-algebra axioms, and the third property is Schröder's Proposition 14.3.

($\Leftarrow$) Again, the first two properties are in the σ-algebra axioms; we need to show that the intersection closure property implies the union closure property. But the logic of Proposition 14.3 holds without loss of generality: $\bigcap_i A_i \in \Sigma$ implies $\bigcap_i M \backslash A_i \in \Sigma$ (by complement closure) implies $M \backslash \bigcup_i A_i \in \Sigma$ (by DeMorgan's law) implies $\bigcup_i A_i \in \Sigma$ (by complement closure again).

14-8. We want to show that $\mu(A) - \mu(B) = \mu(A \backslash B) - \mu(B \backslash A)$. Observe that $A = A \backslash B \sqcup (A \cap B)$ and $B = B \backslash A \sqcup (A \cap B)$. So by countable additivity, $\mu(A) - \mu(B) = \mu(A \backslash B) + \mu(A \cap B) - (\mu(B \backslash A) + \mu(A \cap B)) = \mu(A \backslash B) - \mu(B \backslash A)$.

14-11. *Theorem.* The outer Lebesgue measure on $\mathbb{R}^n$, $\lambda(S) := \inf \left\{ \sum_{j=1}^\infty |B_j| : S \subseteq \bigcup_{j=1}^\infty B_j \right\}$ (for open boxes B_j), is an outer measure.

Proof. (Empty set.) $\lambda(\emptyset) = 0$ because the empty set is a subset of anything, so the infimum of the sizes of collections of boxes that cover the empty set is zero.

(Containment, following Schröder's proof of Theorem 8.6 as hinted.) If $A \subseteq B$, then any collection of boxes that covers B also covers A, so $\left\{\sum_{j=1}^{\infty} |C_j| : B \subseteq \bigcup_{j=1}^{\infty} C_j\right\} \subseteq \left\{\sum_{j=1}^{\infty} |C_j| : A \subseteq \bigcup_{j=1}^{\infty} C_j\right\}$ (the containment goes the other way because if not every point of B is a point of A, then there are coverings of A that don't cover B), so $\lambda(A) \leq \lambda(B)$ (taking a superset can decrease but not increase the greatest lower bound).

(Countable subadditivity, still following Theorem 8.6.) Fix ε. For each $n \in \mathbb{N}$, find a countable set of open boxes $\{B_{n,j}\}_{j=1}^{\infty}$ that cover A_n with $\sum_{j=1}^{\infty} |B_{n,j}| \leq \lambda(A_n) + \frac{\varepsilon}{2^n}$. (The reason we can do that is because $\lambda(A_n)$ is the infimum of the set of box covers, so we can find a cover "just a little bit" bigger.) Then the measure of the union is less than or equal to the measure of the boxes that cover it, $\lambda(\bigcup_n A_n) \leq \sum_n \lambda(A_n) + \frac{\varepsilon}{2^n} = \sum_n \lambda(A_n) + \varepsilon$.

14-12. We want to show that $\mu(T) \leq \mu(S \cap T) + \mu(S^{\complement} \cap T)$. We reason that $\mu(T) = \mu((S \cap T) \cup (S^{\complement} \cap T))$, which is less than $\mu(S \cap T) + \mu(S^{\complement} \cap T)$ by subadditivity.

14-13. *Theorem.* If $\mu(S) = 0$ for outer measure μ, then S is μ-measurable.

Proof. Consider arbitrary T. $(S \cap T) \subseteq S$, so by monotonicity $\mu(S \cap T) \leq \mu(S) = 0$.

$T = (S \cap T) \cup (S^{\complement} \cap T)$, so by subadditivity, $\mu(T) = \mu((S \cap T) \cup (S^{\complement} \cap T)) \leq \mu(S \cap T) + \mu(S^{\complement} \cap T)$. So $\mu(T) \leq \mu(S^{\complement} \cap T)$. But also, $\mu(S^{\complement} \cap T) \leq \mu(T)$ by monotonicity. So $\mu(T) = \mu(S^{\complement} \cap T)$. So $\mu(S \cap T) + \mu(S^{\complement} \cap T) = 0 + \mu(T) = \mu(T)$.

14-14. *Theorem.* If A and B are μ-measurable for outer measure μ, then A∩B is, too.

Proof (mimicking Lemma 9.8 as hinted). Take a test set T. $\mu((A \cap B) \cap T) + \mu((A \cap B)^{\complement} \cap T) = \mu(A \cap B \cap T) + \mu((A^{\complement} \cup B^{\complement}) \cap T)$. Then (as in the proof of Lemma 9.8), we use $(A^{\complement} \cup B^{\complement}) \cap T)$ itself as a test set against A, to get $\mu(A \cap B \cap T) + \mu(A \cap (A^{\complement} \cup B^{\complement}) \cap T) + \mu(A^{\complement} \cap (A^{\complement} \cup B^{\complement}) \cap T)$. Then the intersection of a set and the union of its complement with anything is the intersection of the set and the thing, and the intersection of a set with the union of itself and anything is just the set (because any points in the thing not already in the set get shaved away by the intersection), so we have $\mu(A \cap B \cap T) + \mu(A \cap B^{\complement} \cap T) + \mu(A^{\complement} \cap T)$. By the measurability of B and A, respectively, we can "un-test" to $\mu(A \cap T) + \mu(A^{\complement} \cap T)$ and again to $\mu(T)$.

14-15. *Theorem.* For a sequence of disjoint μ-measurable sets $\{A_j\}_{j=1}^{\infty}$ for outer measure μ, $\bigcup_j A_j$ is μ-measurable, and for test set T, we have $\mu(T) = \sum_{j=1}^{\infty} \mu(A_j \cap T) + \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^{\complement} \cap T\right)$.

Proof. We know that the individual A_j are μ-measurable, *i.e.*, that $\forall j\ \mu(T) = \mu(T \cap A_j) + \mu(T \cap A_j^{\complement})$. To extend that to μ-measurability of the union, we probably need to invoke countable subadditivity somehow? $\bigcup_j A_j$ will be μ-measurable if $\mu(T) = \mu(T \cap (\bigcup_j A_j)) + \mu(T \cap (\bigcup_j A_j)^{\complement})$. $\mu(T \cap (\bigcup_j A_j)) = \mu(\bigcup_j (T \cap A_j)) \leq \sum_{j=1}^{\infty} \mu(T \cap A_j)$ by countable subadditivity. ... okay, I don't see where to take this; let's look at the hinted Lemma 9.9. It's mostly an induction.

(Base.) $\mu(T) = \mu(A_1 \cap T) + \mu(A_1^{\complement} \cap T)$ by the measurability of A_1.

(Induction.) Suppose $\mu(T) = \sum_{j=1}^{n} \mu(A_j \cap T) + \mu(\left(\bigcup_{j=1}^{n} A_j\right)^{\complement} \cap T)$. We want to push this to $n+1$. As in Lemma 9.9, we "push in" the complement with De Morgan's law $(\mu(\left(\bigcup_{j=1}^{n} A_j\right)^{\complement} \cap T) = \mu(\left(\bigcap_{j=1}^{n} A_j^{\complement}\right) \cap T))$ and use the fact that A_{n+1} is measurable $(\mu(\left(\bigcap_{j=1}^{n} A_j^{\complement}\right) \cap T) = \mu(A_{n+1} \cap \left(\bigcap_{j=1}^{n} A_j^{\complement}\right) \cap T) + \mu(A_{n+1}^{\complement} \cap \left(\bigcap_{j=1}^{n} A_j^{\complement}\right) \cap T)$, where the intersection of complements is going to have empty intersection with disjoint A_{n+1} in the first term and the grand intersection in the second term is going to go to $n+1$).

But then we can unify that $\mu(A_{n+1} \cap T)$ with the earlier sum, so we have $\mu(T) = \sum_{j=1}^{n+1} \mu(A_j \cap T) + \mu(\bigcap_{j=1}^{n+1} A_j^{\complement} \cap T)$, induction complete.

At this point I was confused why Lemma 9.9 didn't end there. Talking to OpenAI o1, it seems that the issue is that the induction established an equality for any finite n, but we need to do a little more work to secure the infinite union? Maybe I can move on for now and hope to understand this better later.

14-16. *Theorem.* Consider M a set, $\mu : \mathcal{P}(M) \to [0, \infty]$ an outer measure, and Σ_μ the set of μ-measurable sets. $(M,\ \Sigma_\mu,\ \mu)$ is a measure space.

Proof. (We are hinted to mimic Theorem 9.10 to show that Σ_μ is a σ-algebra and mimic 9.11 to show that μ is countably additive on Σ_μ.)

(Empty set.) $\emptyset \in \Sigma_\mu$ because $\mu(\emptyset) = 0$ by the empty-set axiom for outer measure, and we showed above that sets with null outer measure are measurable.

(Complement closure.) Suppose that S is μ-measurable. We want to show that $S^{\complement} = M \backslash S$ is μ-measurable. What it means for S to be measurable is that for any $T \subseteq M$, $\mu(T) = \mu(S \cap T) + \mu(S^{\complement} \cap T)$. But this is saying the same thing about $S^{\complement}$ (swapping the roles of S and $S^{\complement}$; it's the same statement).

(Countable union closure.) We want to show that the countable union of μ-measurable sets is μ-measurable. Lemma 14.24 (which we tried to prove above in 14-15) shows that this is true for the countable union of disjoint sets. But we can "disjointify" an arbitrary countable union (compare 14-6 above) by inductively defining B_i as the points of A_i that haven't already appeared in $\bigcup_{j<i} A_j$.

(Countable additivity of the measure.) We have a lemma above that says $\mu(T) = \sum_{j=1}^{\infty} \mu(A_j \cap T) + \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^{\complement} \cap T\right)$. The proof of Theorem 9.11 is saying that we're going to use the union *as the test set*: $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{k=1}^{\infty} \mu(A_k \cap \left(\bigcup_{j=1}^{\infty} A_j\right)) + \mu\left((\bigcup_{k=1}^{\infty} A_k)^{\complement} \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right)$. Then the summands become A_k (because intersecting with the union of the other A_j does nothing) and the second term vanishes (because the intersection of a set and its complement is empty), leaving us with $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$.

14-17. We want to show that outer Lebesgue measure is the same if you use dyadic rational ($\frac{n}{2^k}$, $n, k \in \mathbb{Z}$) open boxes rather than generic open boxes.

Intuitively, our proof is going to rely on the product of dyadic rationals $\{\frac{k}{2^n} : n, k \in \mathbb{Z}\}^n$ being dense in $\mathbb{R}^n$: anything you can do with open boxes, you can get arbitrarily close to with dyadic open boxes.

Suppose not: $\lambda(S) = \inf\left\{\sum_{j=1}^{\infty} |B_j| : S \subseteq \bigcup_{j=1}^{\infty} B_j,\ B_j \text{ open box}\right\} \neq \inf\left\{\sum_{j=1}^{\infty} |B_j| : S \subseteq \bigcup_{j=1}^{\infty} B_j\ , B_j \text{ open dyadic box}\right\}$.

Fix ε. We know that for a measurable set S, there exist open boxes B_j that cover S such that $|\lambda(S) - \sum_j |B_j|| < \frac{\varepsilon}{2}$.

Then we want to cover those boxes with dyadic open boxes D_j such that $\left|\sum_j |B_j| - |D_j|\right| < \frac{\varepsilon}{2}$. The thing that makes this tricky is that there can be countably many boxes, so to calculate how to share our $\frac{\varepsilon}{2}$ margin over n dimensions of countably rather than finitely many boxes, we're going to need some kind of "series trick": say, allocate $\frac{1}{2^j}$ of our $\frac{\varepsilon}{2}$ error budget to the jth box, where it will be shared among the n dimensions of the box—but we need to take into account that the errors along different dimensions combine multiplicatively. For example, if $B_i = (0,1) \times (0,1)$ and we approximate it with $D_i = (-\frac{1}{2^{10}}, 1 + \frac{1}{2^{10}}) \times (-\frac{1}{2^{10}}, 1 + \frac{1}{2^{10}})$, the error $|D_i - B_i| = (1 + \frac{2}{2^{10}})^2 - 1^2 = (1 + \frac{1}{2^9})^2 - 1$. The algebra here is quickly getting messy, though: the difference-of-nth powers formula involves a summation. (OpenAI o1 is agreeing that this is the right idea, and claims that there's a mean-value-theorem/partial-derivatives approach to nailing down the inequality, which I'm not going to allocate time to now.)

14-18. We want to show that for $A,\ B \subseteq \mathbb{R}$, $\lambda(A) = 0$ (the Lebesgue measure on $\mathbb{R}$), $\lambda(A \times B) = 0$ (the Lebesgue measure on $\mathbb{R}^2$). When we say that $\lambda(A)$, what we mean is that

$$\inf\left\{\sum_{j=1}^{\infty}(a_{j,2} - a_{j,1}) : A \subseteq \bigcup_{j=1}^{\infty}(a_{j,1} - b_{j,2})\right\} = 0$$

That means that for any ε, there exists a sequence of open intervals that cover A whose union has length less than ε.

Take an arbitary ε. Choose a corresponding sequence of open intervals $(a_{j,1},\, a_{j,2})$ to cover A whose union has length less than $\frac{\varepsilon}{\lambda(B)}$, and consider a sequence of open intervals $(b_{j,1}, b_{j,2})$ that cover B such that for all j, $b_{j,2} - b_{j,1} \leq \lambda(B)$. Then $(a_{j,1}, a_{j,2})(b_{j,1}, b_{j,2})$ cover $A \times B$ and

$$\sum_j (a_{j,2} - a_{j,1})(b_{j,2} - b_{j,1}) \leq \lambda(B) \sum_j (a_{j,2} - a_{j,1}) < \lambda(B) \frac{\varepsilon}{\lambda(B)} = \varepsilon$$

Therefore $\lambda(A \times B) = \inf\left\{\sum_{j=1}^{\infty}(a_{j,2} - a_{j,1})(b_{j,2} - b_{j,1}) : A \times B \subseteq \bigcup_{j=1}^{\infty}(a_{j,1}, a_{j,2}) \times (b_{j,1}, b_{j,2})\right\} = 0$.

I was erroneously really proud of this, but Claude 3.5 Sonnet points out it only works as written if $\lambda(B)$ is finite. The reason I did it that way is because I wanted to bound the intervals covering B so that I could pull the bound out of the summation. But actually, any finite bound works (because $\mathbb{R}$ is second-countable), so it was retarded to name $\lambda(B)$ as the bound, because that might not be finite.