Zack M. Davis "Real Analysis II" midterm 17 october 2024 32.5/40

(1) The Weierstrass M-test states that for a sequence $\{M_k\}$ & sequence of functions $\{f_k(x)\}$ where $M_k \geq 0$, if $\sum_k M_k$ converges & $|f_k(x)| \leq M_k$, then $\{f_k(x)\}$ converges absolutely & uniformly.

A series of functions converges uniformly on a set $E$ to $f$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall x \in E$, $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$, that is, $N$ doesn't depend on $x$.

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Dirichlet's test states that if the partial sums $\sum_{k=1}^{n} a_k$ are bounded & $b_k$ goes to zero nonincreasingly, then $\sum_k a_k b_k$ converges.

(2) (a) $\boxed{\text{False.}}$ Suppose without loss of generality that $a_k b_k$ are rational numbers $\frac{c_k}{d_k}$, where $c_k$ & $d_k$ are integers. Then $\sum_k a_k := \sum_k c_k$ is an infinite series of integers, which diverges.

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(b) This is the converse of the ratio test. I think it's $\boxed{\text{true}}$ (& that when the ratio test limit is 1, the series either diverges or is conditionally convergent), but we need a stronger justification... suppose for a contradiction that $\sum a_k$ converges & $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \geq 1$. But then the terms would not be decreasing, so $\sum a_k$ would diverge by the divergence test. Contradiction! NOT NEC. WHAT IF $a_k = \frac{1}{k^2}$?

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(c) $\boxed{\text{True.}}$ Split $a_k$ into terms $\geq 1$, & those less than 1. $\sum_{a_k \geq 1} a_k$ has finitely many terms, so it trivially converges & so does $\sum_{a_k \geq 1} a_k^2$.

For the $a_k$ less than one, $a_k^2 < a_k$, so $\sum_{a_k < 1} a_k^2$ converges by the comparison test.

✓ (d) $\boxed{\text{False.}}$ Consider $a_k = \frac{1}{k}$. $\sum_k \frac{1}{k}(1)^k = \sum_k \frac{1}{k}$, & the harmonic series is known to diverge.

(3) ~~$\limsup_{n\to\infty} \left|\frac{n(x+2)^n}{5^{n-1}}\right|^{1/n} = \limsup_{n\to\infty} \frac{n^{1/n}(x+2)}{5^{\frac{n-1}{n}}}$~~ ... wait, no, the root test-like radius of convergence calculation is $\limsup |a_k|^{1/k}$, not $\limsup |a_k(x-x_0)^k|^{1/k}$.

$\limsup_{n\to\infty} \left|\frac{n}{5^{n-1}}\right|^{1/n} = \limsup_{n\to\infty} \frac{n^{1/n}}{5^{\frac{n-1}{n}}}$ goes to $1/5$, suggesting a radius of convergence of 5.

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& the series is centered at $-2$. To find the exact interval of convergence, we need to check the endpoints, $-2-5=-7$ & $-2+5=3$.

~~$\sum_n \frac{n5^n}{5^{n-1}} = \sum_n 5n$, which diverges~~ $\sum_n \frac{n\,5^n}{5^{n-1}} = \sum_n 5n$, which diverges.

$\sum_n \frac{n(-5)^n}{5^{n-1}} = \sum_n \frac{n(-1)^n 5^n}{5^{n-1}} = \sum_n (-1)^n 5n$, which also diverges.

$\boxed{\text{Thus our radius of convergence is 5 \& our interval of convergence is } (-7, 3).}$

(continued →)

(4) (a) $g(2) = \frac{2}{1+2} = \frac{2}{3}$, so if we can show that $g(x)$ is an increasing function on this domain, that will imply that it's $\leq 2/3$ on this domain.

But $g'(x) = \frac{1}{1+x} + x\frac{d}{dx}(1+x)^{-1} = \frac{1}{1+x} + x\frac{1}{(1+x)^2}$, which is surely positive if $x$ is positive.

$\frac{4}{5}$ THIS FORMULA FOR $g'$ IS WRONG.

(b) Let's try the Weirstrass M-test.? But what should $M_k$ be?

We know that $g(x) \leq 2/3$ (on our domain) & $f_n(x) = (g(x))^n$.

So consider $M_k := (2/3)^k$. $\sum_k M_k$ converges because it's a geometric series.

$$|f_k(x)| = f_k(x) = g(x)^k \leq (2/3)^k = M_k.$$

$\frac{5}{5}$ So the M-test implies that $\sum_n f_n$ converges uniformly on $[1,2]$, which is quod erat demonstrandum.

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