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"Real Analysis II" final exam
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44/50 40/40 + 4/10 #4 NEEDS TO BE CHECKED

(1) (a) A metric space is complete iff every Cauchy sequence converges.

10 (b) A subset A of metric space X is compact if every open cover of A has a finite subcover, $\{C_\alpha\}_\alpha$ s.t. $A \subseteq \bigcup_\alpha C_\alpha$ & $\forall \alpha$ C_α is open being the definition of an open cover.

(d) A set is connected iff it can't be written as a disjoint union of two nonempty open sets, $U \cup V = A$, $U \cap V = \emptyset$, U, V open & $\neq \emptyset$

(2) (a) $\boxed{\text{False.}}$ ✓ Any open interval (a, b) is connected but not compact.

(b) $\boxed{\text{False.}}$ ✓ Consider $E_i := [i, i+1]$. $\forall i$ E_i is compact, but $\bigcup_{i=1}^{\infty} = [1, \infty)$ is unbounded.

10 (c) $\boxed{\text{False.}}$ ✓ Consider the discrete space & $r := 1/2$. For every x, $B_{1/2}(x) = \{x\}$, which is closed because finite sets contain all their limit points.

(d) $\boxed{\text{False.}}$ ✓ Openness is preserved by continuous <u>inverse</u> images, not ("forward") images. But we should show a counterexample... a constant function will do. $f(x) = 3$ yields $f((0,1)) = \{3\}$, which is closed.

(3) (a) ✓ The inverse image of an open set is open iff the function is continuous.

10 (b) ✓ Every set in $\mathbb{R}_d$ is open, because for all $E \subseteq \mathbb{R}_d$, for all $x \in E$, $B_{1/2}(x) = \{x\} \subseteq E$. If every set is open, then the inverse image of any set in the codomain is open, so all functions are continuous. <u>quod erat demonstrandum</u>

(4) Because $\{x_n\}$ converges, $\forall \varepsilon$ $\exists N$ $n \geq N$ implies $d(x_n, x_0) < \varepsilon$. DO YOU MEAN x_0?

Consider $C_j := B_{1/j}(x)$ for $j \in \mathbb{N}^+$. Then $\{C_j\}_{j=1}^{\infty} \cup \{x_k : d(x_k, x_0) \geq 1\}$ is an open cover of of $\{x_n\}_n \cup \{x_0\}$. ($\{x_k : d(x_k, x_0) \geq 1\}$ can be relatively open in $\{x_n\}_n \cup \{x_0\}$ even if it's closed in X.) This open cover has no finite subcover: suppose for a contradiction that it did. Then there would be a largest j for which C_j is in the subcover. Then the putative subcover would not contain points y such that $d(y, x_0) < 1/j+1$, which we know exist because $\{x_n\}$ converges to x_0. So our subcover does not cover the set. Contradiction!

(continued on other side →)

(5) (a) X is not connected iff it can be written as $U \cup V$ for open nonempty sets U & V with $U \cap V = \emptyset$.

(b) $f^{-1}(0)$ & $f^{-1}(1)$ will be the connected components of our disconnection.

We know that $f^{-1}(0) \cup f^{-1}(1) = X$ because every point in the domain must be mapped into a point in the codomain.

We know that $f^{-1}(0)$ & $f^{-1}(1)$ are nonempty because f is surjective ("onto").

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We know that $f^{-1}(0)$ & $f^{-1}(1)$ are disjoint because a function can't map a point to more than one value; $x \in f^{-1}(0) \cap f^{-1}(1)$ would imply $f(x)=0$ & $f(x)=1$.

We still need to show that $f^{-1}(0)$ & $f^{-1}(1)$ are open.

We know that $\forall x \in X \; \forall \varepsilon \; \exists \delta \; \forall y \in X \; d(x,y) < \delta \rightarrow d(f(x), f(y)) < \varepsilon$.

Consider $x \in f^{-1}(0)$ (without loss of generality, $f^{-1}(1)$), & $\varepsilon := 1/2$.

Then there exists δ such that $d(x,y) < \delta$ implies $f(y) = 0$ (because the only way $f(y)$ can be within $1/2$ of $f(x) = 0$ in $\{0,1\}$ is by also being 0).

So $B_\delta(x) \subseteq f^{-1}(0)$, so $f^{-1}(0)$ is open.

<u>quod erat demonstrandum</u>