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"Measure & Integration" midterm
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(1) $|E| = 0$ means that $\inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k\right\} = 0$.

That is, for all ε, there exists a collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \ell(I_k) < \varepsilon$ & $E \subseteq \bigcup_{k=1}^{\infty} I_k$. (because $\sum_{k=1}^{\infty} \ell(I_k) < |E| + \varepsilon$, but $|E| = 0$).

Intuitively, then, $\mathbb{R} \setminus E$ has "full measure" ∞, & so every interval I must intersect $\mathbb{R} \setminus E$.

In more detail, every open interval I has positive measure; $|I| > 0 = |E|$.
The monotonicity property of outer measure states that $A \subseteq B$ implies $|B| \geq |A|$.
We know that $|I| > |E|$, so it cannot be the case that $I \subseteq E$, by
10 *modus tollens*. But if $I \not\subseteq E$, then $I \cap E^c = I \cap \mathbb{R} \setminus E \neq \emptyset$,
which is quod erat demonstrandum.

(2) Because we have an increasing sequence of non-negative measurable functions, we can invoke the monotone convergence theorem to "switch" the limit & integral signs, & we have

$\int \lim_{n\to\infty} \mathbb{1}_{[0,n]} \frac{n}{x+n} \, d\lambda$ But $\lim_{n\to\infty} \mathbb{1}_{[0,n]} = 1$ & $\lim_{n\to\infty} \frac{n}{x+n} = \frac{1}{x}$, so we have

$\int \frac{1}{x} \, d\lambda = \log x + C$

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SHOULD BE $\lambda_{[0,t]}$

(3) (a) $|E| = \inf \{ \sum_{k=1}^{\infty} \ell(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k \}$. means that there exists a collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \ell(I_k) < |E| + \varepsilon$ for any $\varepsilon > 0$.
Take $U := \bigcup_{k=1}^{\infty} I_k$. U is open because the union of open intervals is open, & we have $|E| \le |U| \le |E| + \varepsilon$.

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(b) From the above with $\varepsilon := 1/n$, for every $n \in \mathbb{N}$, we can find a U_n such that $|E| \le |U_n| \le |E| + 1/n$.
But we have a theorem about the measure of a decreasing intersection:
if $|U_1| < \infty$, then $|\bigcap_{n=1}^{\infty} U_n| = \lim_{n\to\infty} |U_n|$. $\lim_{n\to\infty} \frac{1}{n} = 0$, so we have
$|E| \le \lim_{n\to\infty} |U_n| = |\bigcap_{n=1}^{\infty} U_n| \le |E|$, so $|\bigcap_{n=1}^{\infty} U_n| = |E|$.
Also, for all n, $E \subseteq U_n$, so $E \subseteq \bigcup_{n=1}^{\infty} U_n$ as well.

HOW DO YOU KNOW $U_1 \supseteq U_2 \supseteq \ldots$?

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(4) $\int 1_E f \, d\mu = \sup \{ \sum_{k=1}^{\infty} \mu(A_k) \inf_{x\in A_k} 1_E f(x) : \{A_k\} \text{ a partition of } S \} = 0$

Suppose for a contradiction that $\mu(\{x \in X : f(x) \neq 0\}) \neq 0$, that is, that there exists a set of positive measure on which f takes nonzero values.
Then there must exist some n for which $A_n := \{x \in X : f(x) \ge 1/n\}$ has positive measure, $\mu(A_n) > 0$...
Rather, consider the partition $B_n := \{x \in X : \frac{1}{n+1} < f(x) \le \frac{1}{n}\}$.
By our hypothesis above, there exists an n such that $\mu(B_n) > 0$.
Call it n_0.
Then $\mu(B_{n_0}) \inf_{x\in B_{n_0}} 1_E f(x) = \mu(B_{n_0}) \frac{1}{n_0} > 0$, so $\sum_{k=1}^{\infty} \mu(B_k) \inf_{x\in B_k} 1_E B_k > 0$.
The existence of a partition $\{B_k\}$ of S such that $\sum_{k=1}^{\infty} \mu(B_k) \inf_{x\in B_k} 1_E f > 0$ shows that 0 is not an upper bound of such sums, thus not the least upper bound, thus $\int 1_E f \, d\mu > 0$. Contradiction!

WHAT IS E?

THIS IS NOT A PARTITION. IT'S NOT FINITE.

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