Zack M. Davis
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"Measure & Integration" final   80%??   NO!! 48/50 !!!

① ⓐ 5 <u>Dominated convergence theorem.</u> If $\{f_k\}_{k=1}^{\infty}$ is a pointwise convergent sequence of functions on $[0,1]$ & there exists a function $g(x)$ such that for all $k \in \mathbb{N}$ & all $x \in [0,1]$, $|f_k(x)| \leq |g(x)|$, then
$$\lim_{k\to\infty} \int_0^1 f_k(x)\, d\mu = \int_0^1 \lim_{k\to\infty} f_k(x)\, d\mu.$$

10 ⓑ $\cos(x^n)$ is dominated by 1 (i.e., $|\cos(x^n)| \leq 1$), so the hypotheses of the dominated convergence theorem are fulfilled.

5 $$\lim_{n\to\infty} \int_0^1 \cos(x^n)\, d\mu = \int_0^1 \lim_{n\to\infty} \cos(x^n)\, d\mu \quad \text{But } \lim_{n\to\infty} x^n = \begin{cases} 0 & x<1 \\ 1 & x=1 \end{cases}$$

So we have $\int_0^1 \cos(0) = \int_0^1 1 = 1$

② ⓐ $f \in L^1([0,1])$ means that $\int_0^1 f\, d\mu < \infty$

10 We want $\int_0^1 x^k f(x) < \infty$. Let's try integration by parts? $\int u\, dv = uv - \int v\, du$

$u := x^k \qquad v = F(x) \qquad x^k F(x)\Big|_0^1 - \int_0^1 F(x)\, k x^{k-1}$

5 $du = kx^{k-1} \qquad dv = f(x)$

where I'm assuming $F(x)$ is an antiderivative of f...

Actually, wait. I think this is the wrong approach — instead of directly trying to evaluate $\int_0^1 x^k f(x)$ to show that it's finite, we should show that it's dominated by something integrable. For all $x \in [0,1]$ & all $k \in \mathbb{N}$, $|x^k| \leq 1$, so $|x^k f(x)| \leq |f(x)|$. A function dominated by a L^1 function is itself L^1.

ⓑ From part a, we can invoke the dominated convergence theorem.

5 $$\lim_{k\to\infty} \int_0^1 x^k f(x) = \int_0^1 \lim_{k\to\infty} x^k f(x) = 0 \quad \left(\text{recalling again that } \lim_{k\to\infty} x^k = \begin{cases} 0 & x<1 \\ 1 & x=1 \end{cases}\right)$$

③ $Tf(x) = f(|x|)$ for T linear $L^1 \to L^1$

6/6 ⓐ We want to show that T is bounded, & we are given the hint that $\int_{-\infty}^0 h(x)\, dx = \int_0^\infty h(-x)\, dx$. I guess that implies $\|f\| = \int_{-\infty}^{\infty} |f|\, d\mu$ is the norm — maybe that was already implied by L^1? YES

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$$\|f(|x|)\| = \int_{-\infty}^{\infty} f(|x|)\, d\mu = \int_{-\infty}^{0} f(|x|)\, d\mu + \int_0^\infty f(|x|)\, d\mu$$
$$= \int_0^\infty f(|-x|)\, d\mu + \int_0^\infty f(|x|)\, d\mu \quad \text{(by the hint)}$$
$$= 2\int_0^\infty f(|x|)\, d\mu < \infty \quad \text{(because we know that } f \in L^1\text{)}$$

2/4 ⓑ $\|T\| = 2$ from the above   NO, YOU'VE ONLY SHOWN $\|T\| \leq 2$.

④ The Hahn-Banach theorem states that a bounded linear functional can be extended without increasing its norm.
Let $f, g \in V$ with $f \neq g$. The dual space V' is the set of linear functionals from V to its field. Defining a functional $\phi \in V'$ with $\phi(f) \neq \phi(g)$ would seem to be trivial? — for example, $\phi(f) = 0$ & $\phi(g) = 1$. Oh, I see — what makes this nontrivial is ensuring the existence of a linear functional.

Let $\{u_\alpha\}$ for $\alpha \in A$ be a basis for V.
Then f & g can both be expressed in terms of the basis:
$f := \sum_\alpha c_\alpha u_\alpha$ & $g := \sum_\alpha d_\alpha u_\alpha$.
Because $f \neq g$, there exists some α_0 such that $c_{\alpha_0} \neq d_{\alpha_0}$.

WHY IS IT LINEAR? IS IT BOUNDED?

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Consider $\phi(f) := c_{\alpha_0} u_{\alpha_0}$ (i.e., a functional that extracts the u_α coordinate; we also have $\phi(g) = d_{\alpha_0} u_{\alpha_0}$). Thus $\phi(f) \neq \phi(g)$.
By the Hahn-Banach theorem, ϕ (which one might construe as existing on the subspace spanned by u_{α_0}) can be extended to all of V', which is quod erat demonstrandum.

THERE'S AN EASIER WAY, DEFINE $\phi : \text{span}(f-g) \to F$, $\alpha(f-g) \mapsto \alpha$. IT'S LINEAR AND BOUNDED, THEN EXTEND BY H-B.

⑤ Suppose not: that there exists ε such that for all N,
$\neg(n \geq N \to |\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\}| < \varepsilon)$
$\neg(n < N \vee |\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\}| < \varepsilon)$
$\exists\ n \geq N$ & $|\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\}| \geq \varepsilon$
We know that $\|f_n - f\| \to 0$, which means that
$\forall \varepsilon\ \exists n \int_{-\infty}^{\infty} |f_n - f| < \varepsilon$.

Suppose for a contradiction that there exists ε such that for all N, there exists $n \geq N$ such that $|\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\}| \geq \varepsilon$.
Let $E := \{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\}$ for arbitrary ε, & such corresponding n.

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~~$\int_{-\infty}^{\infty} |f_n - f| \, dx \geq \int_E |f_n - f| \, d\mu \leq |E| \sup |f_n - f|$~~ Then

$$\int_E |f_n - f| \, d\mu \geq \mu(E) \cdot \inf_{x \in E} |f_n(x) - f(x)| = \varepsilon^2$$

But $\int_{\mathbb{R}} |f_n - f| \, d\mu \geq \int_E |f_n - f| \, d\mu$, so $\int_{\mathbb{R}} |f_n - f| \, d\mu > \varepsilon^2$, but earlier we had supposed that $\int_{\mathbb{R}} |f_n - f| < \varepsilon$. Contradiction!