Pinpointing the Generalized Factorial

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1 Prelude: A Historical Accident

Driven ceaselessly by a compulsion to generalize, students of mathematics often wonder whether there exists an analogue of the factorial function $n! := \prod_{k=1}^{n} k$ that works for non-integers. The question is traditionally answered in the affirmative by presenting a function Γ defined on $\mathbb{R}$ (and indeed $\mathbb{C}$) for which $\Gamma(x) = (x-1)!$.

This reply prompts an obvious followup query: "Thanks, but what's with the shift by one? This function is generalizing the factorial. You should define it such that $\Gamma(x) = x!$, right?"

In the present author's research, no one seems to have a good answer to this. It seems like Adrien-Marie Legendre chose $\Gamma(n) = (n-1)!$ back in the early 19th century and it somehow caught on, even though Carl Friedrich Gauss was contemporaneously working with the same function (up to the shift) as $\Pi(n) := n!$.

Some authors try to rationalize Legendre's blunder by arguing that it's better to have the rightmost singularity at 0 than -1, or that we should integrate over the Haar measure.[1]

The present author doesn't buy it. In the spirit of efforts to promote $\tau := 2\pi$ rather than π as the correct circle constant in accordance with logic and in opposition to blind historical inertia,[2] this report concerns itself with the true generalized factorial Π rather than the treacherous so-called "gamma function" Γ.

2 The Only Reasonable Way To Do It

We want a continuous function $\Pi(x)$ that generalizes the factorial: how might we go about engineering such a function?

2.1 The Obvious First Guess (Which Is Wrong)

We want $\Pi(n) = n!$ for $n \in \mathbb{N}$, a function that interpolates the factorials. We're used to approximating functions with power series, which are limits of polynomials. There is then a natural temptation to consider polynomial interpolations of $(k, k!)$ for $k \in \{0...n\}$ and take the limit $n \to \infty$.

It turns out that this doesn't work. Our standard technique for interpolating points with a polynomial is due to Lagrange. Suppose we want to interpolate the points (x_k, y_k) for $k \in \{0...n\}$. We define the Lagrange basis element

$$\ell_k(x) := \prod_{\substack{0 \le j \le n \\ j \ne k}} \frac{x - x_j}{x_k - x_j}$$

And then

$$P_n(x) := \sum_k y_k \ell_k(x)$$

The idea is that when $x := x_k$, then for all $j \ne k$, ℓ_j contains a factor of $x_j - x_j = 0$ so that the $y_j \ell_j(x_k)$ term vanishes, and $\ell_k(x_k) = \prod_{j \ne k} \frac{x_k - x_j}{x_k - x_j} = 1$, engineering the outcome $P_n(x_k) = y_k$.

If we apply this method to $(x_k, y_k) := (k, k!)$, we get

$$\text{“}\Pi\text{”}_n(x) := \sum_{k=0}^{n} k! \prod_{\substack{0 \le j \le n \\ j \ne k}} \frac{x - j}{k - j}$$

[1] *https://mathoverflow.net/questions/20960/why-is-the-gamma-function-shifted-from-the-factorial-by-1*
[2] *https://www.tauday.com/tau-manifesto*

The naïve hope is that

$$\text{“}\Pi\text{”}(x) := \lim_{n\to\infty} \sum_{k=0}^{n} k! \prod_{\substack{0\le j\le n\\ j\ne k}} \frac{x-j}{k-j}$$

would be our generalized factorial. But it doesn't work. Our finite polynomial interpolations depended on the $\ell_j(k)$ terms vanishing, but when $x \notin \mathbb{N}$, none of the terms vanish: with a factor of $k!$ on each, the series doesn't come close to converging.

2.2 Using the Recursive Desideratum

Brute empiricism has failed us. A more promising approach might be to work with the desideratum

$$\Pi(x+1) = (x+1)\Pi(x) \tag{1}$$

(Note that this is different from the analogous functional equation for the treacherous so-called "gamma" function, $\Gamma(x+1) = x\Gamma(x)$.)

But what to do with it?

As an inspired hack, consider integration by parts—a procedure known to sometimes transform an expression into a "version of itself" "altered" by a multiplicative factor (as in "rapid repeated integration by parts"). The integration by parts formula is:

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$$

The hope here is that there exist choices of u, v, a, and b that end up satisfying (1):

$$\underbrace{\int_a^b u\,dv}_{\Pi(x+1)} = uv\big|_a^b - \underbrace{\int_a^b v\,du}_{(x+1)\Pi(x)}$$

(We'll worry about the uv term in a moment.)

A period of contemplation should convince the reader that it suffices to take $dv = -v$ and (construing u as taking an argument x different from the one that d indicates differentiation by) $du(x+1) = (x+1)\cdot u(x)$. Disciples of the differential and integral calculus who know that the natural exponential is its own derivative and integral can use that fact in conjunction with the chain rule to infer that (taking the variable we *are* differentiating by as t) if v is not zero, then we must have $v := \exp -t$. Such disciples will probably also successfully pattern-match $du(x+1) = (x+1)\cdot u(x)$ to the power rule: $\frac{d}{dt}t^{x+1} = (x+1)\cdot t^x$ and conclude that $u := t^{x+1}$, although it make take some discipline to accept x being a constant concerning differentiation with respect to t.

At this juncture, we have:

$$\int_a^b t^{x+1}\exp -t\,dt = t^{x+1}\exp -t\big|_{t=a}^b + \int_a^b (x+1)t^x \exp -t\,dt$$

Where it remains to worry about the uv term, and settle on a choice of a and b. These problems solve each other. Instead of a and b, let's use the symbols ε and R (respectively). The perceptive reader can probably tell from the choice of symbols where I'm going with this: the $t^{x+1}\exp -t\big|_{t=\varepsilon}^R$ terms cancel in the limit:

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty} R^{x+1}\exp(-R) - \varepsilon^{x+1}\exp(-\varepsilon)$$

(Because $\exp(-R)$ goes to zero faster than R^{x+1} goes to infinity, and ε^{x+1} goes to zero while $\exp(-\varepsilon)$ goes to 1.)

To recap, we've just found that

$$\int_0^\infty t^{x+1}\exp(-t)\,dt = (x+1)\int_0^\infty t^x \exp(-t)\,dt$$

such that (1) can be satisfied by the function defined by an improper integral:

$$\Pi(x) := \int_0^\infty t^x \exp(-t)\,dt \tag{2}$$

And $\Pi(0) = \int_0^\infty t^0 \exp(-t)\,dt = \int_0^\infty \exp(-t)\,dt = \lim_{R\to\infty} \exp 0 - \exp(-R) = 1 - 0 = 1$, according with $0! = 1$. Thus, we find that (2) is truly a generalized factorial function.

2.3 Log Convexity and the Bohr–Mollerup Characterization

We've just shown that $\Pi(x) := \int_0^\infty t^x \exp(-t)\,dt$ is a generalized factorial function. But is it *the* generalized factorial? One could answer: No, there are infinitely many other functions that interpolate the points $(k, k!)$: for example, $\Pi(x) + \sin(\tau x)$ will do. (Hidebound reactionaries would say $\Gamma(x) + \sin(2\pi x)$.) But that seems like "cheating."

A bolder answer might be Yes—there might be an infinitude of functions passing through the points $(k, k!)$, but $\Pi(x)$ is the only "good" one.

Explaining the sense of "good" that uniquely pinpoints $\Pi(x)$ will require some preparation and definitions—some doubtlessly familiar to the reader.

2.3.1 (Log) Convexity Facts!

A function f is said to be *convex* iff for all $t \in [0,1]$ and for all x_1 and x_2 in the domain, $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$. If f is twice-differentiable, then its second derivative is positive.

A consequence is *midpoint convexity* (just choose $t = \frac{1}{2}$):

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}(f(x_1) + f(x_2))$$

By an induction-like argument, midpoint convexity turns out to generalize to an arbitrary finite number of points:

$$f\left(\frac{\sum_k x_k}{n}\right) \leq \frac{1}{n}\sum_{k=1}^{n} f(x_k) \tag{3}$$

Functions satisfying (3) are said to be *weakly convex* by authors such as Artin,[3] although it's really not *much* weaker. (The only weakly convex but non-convex functions are non–Lebesgue-measurable monstrosities constructed using a Hamel basis for $\mathbb{R}$.[4] The present author finds it hard to care less.)

A function f is said to be *logarithmically convex* iff $\log f$ is convex.

I promise we're going somewhere with this.

Proposition. If f is logarithmically convex, then $f\left(\frac{\sum_{k=1}^n x_k}{n}\right)^n \leq \prod_{k=1}^n f(x_k)$.

Proof. "Weak" logarithmic convexity and elementary log properties imply $\log f\left(\frac{\sum_k x_k}{n}\right) \leq \frac{1}{n}\sum_{k=1}^n \log f(x_k) = \frac{1}{n}\log\prod_{k=1}^n f(x_k) = \log\left(\prod_{k=1}^n f(x_k)\right)^{\frac{1}{n}} = \log\sqrt[n]{\prod_{k=1}^n f(x_k)}$; exponentiating yields $f\left(\frac{\sum_k x_k}{n}\right) \leq \sqrt[n]{\prod_{k=1}^n f(x_k)}$ and taking it to the nth yields $f\left(\frac{\sum_k x_k}{n}\right)^n \leq \prod_{k=1}^n f(x_k)$, which is *quod erat demonstrandum.*

Theorem. The sum of logarithmically convex functions is logarithmically convex.
(Left as an exercise to the reader.)

Lemma. If $f(x,t)$ is logarithmically convex in x, then $\int_a^b f(t,x)\,dt$ is logarithmically convex in x.

Proof. For $n \in \mathbb{N}$, let $f_n := \frac{b-a}{n}\sum_{k=0}^{n-1} f(x,\, a + k\left(\frac{b-a}{n}\right))$. Then f_n is logarithmically convex as the sum of logarithmically convex functions (by the previous theorem). But $\lim_{n\to\infty} f_n$ is the Riemann sum for $\int_a^b f(t,x)\,dt$, so that's logarithmically convex, too.

Theorem. $\Pi(x)$ is logarithmically convex.

Proof. As a function of x, $t^x \exp(-t)$ is logarithmically convex, so by the lemma, $\Pi(x) := \int_0^\infty t^x \exp(-t)\,dt$ is, too.

[3] Emil Artin, *The Gamma Function*, Ch. 1, "Convex Functions"
[4] *https://math.stackexchange.com/questions/302684/weak-convexity-and-continuity*

2.3.2 Bohr–Mollerup, and a Product Formula

Having introduced the notion of logarithmic convexity, we are now in a position to present the following

Theorem (Bohr and Mollerup via Artin). If a function f satisfies the following conditions:

1. (Factorial-like recursion.) $f(x+1) = (x+1)f(x)$
2. (Logarithmic convexity.) f is logarithmically convex on $x > -1$.
3. (Initial condition.) $f(0) = 1$

then $f(x) = \Pi(x)$.

Proof. Suppose we have such an f. From factorial-like recursion and the initial condition, we know that $f(n) = \Pi(n) = n!$ for $x \in \mathbb{N}$. If we can show that $f(x) = \Pi(x)$ in an integer interval (*e.g.*, $x \in [0,1]$), then factorial-like recursion will imply that $f(x) = \Pi(x)$ everywhere.

So fix $x \in (0,1]$. For a convex function, difference quotients "get bigger as you go to the right":

$$\frac{\log f(n-1) - \log f(n)}{(n-1)-n} \leq \frac{\log f(n+x) - \log f(n)}{(x+n)-n} \leq \frac{\log f(n+1) - \log f(n)}{(n+1)-n}$$

But at $n \in \mathbb{N}$, $f(n) = n!$:

$$\frac{\log(n-1)! - \log n!}{(n-1)-n} \leq \frac{\log f(n+x) - \log n!}{(n+x)-n} \leq \frac{\log(n+1)! - \log n!}{(n+1)-n}$$

And then we do some more algebra:

$$\frac{\log \frac{(n-1)!}{n!}}{-1} \leq \frac{\log f(x+n) - \log n!}{(x+n)-n} \leq \frac{\log \frac{(n+1)!}{n!}}{1}$$

$$\log(n) \leq \frac{\log f(x+n) - \log n!}{x} \leq \log(n+1)$$

$$x\log(n) \leq \log f(x+n) - \log n! \leq x\log(n+1)$$

$$x\log(n) + \log n! \leq \log f(x+n) \cancel{- \log n! + \log n!} \leq x\log(n+1) + \log n!$$

$$\log(n)^x + \log n! \leq \log f(x+n) \leq \log(n+1)^x + \log n!$$

$$\log n^x n! \leq \log f(x+n) \leq \log(n+1)^x n!$$

The logarithm is monotonic, so we can drop them and keep the inequality:

$$n^x n! \leq f(x+n) \leq (n+1)^x n!$$

But we know from the factorial-like recursion that $f(x+n) = f(x)\prod_{k=x+1}^{x+n} k$, so

$$\frac{n^x n!}{\prod_{k=x+1}^{x+n} k} \leq f(x) \leq \frac{(n+1)^x n!}{\prod_{k=x+1}^{x+n} k}$$

Thus via a "squeezing" argument, we derive a limiting product formula for any function that satisfies our three conditions, the uniqueness of limits surprisingly implying that there is only one such function:

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{\prod_{k=x+1}^{x+n} k}$$

And since our Π function satisfies the three conditions, we must have $f(x) = \Pi(x)$.

3 Conclusion

This has been a hauntingly brief introduction to the generalized factorial (often known as the so-called "gamma function") and why it could not be otherwise in any possible universe. Much more has and remains to be written about its true nature. (For example, the vaunted process of analytic continuation extends the domain of Π to $\mathbb{C}$ uniquely.) But not in this revision of this report.