Midterm

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Abstract

"Take-home midterm" for Prof. Lai's "Theory of Functions of a Complex Variable."

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1. Let

$$f(z) := \cos\left(\frac{1+z}{1-z}\right)$$

Proposition. $f(z)$ is analytic on $|z| < 1$.

Proof. The composition of analytic functions is analytic. $g(z) := \frac{1+z}{1-z}$ is analytic when $|z| < 1$ because the denominator is only zero when $z := -1$ (and $|-1| \not< 1$), cosine is known to be analytic, and $f(z) = \cos(g(z))$.

Proposition. $f(z)$ has infinitely many zeroes in $\mathbb{D}$.

Proof. The equation $0 = \cos\left(\frac{1+z}{1-z}\right)$ implies $\arccos 0 = \frac{\pi}{2} + k\pi = \frac{1+z_k}{1-z_k}$ for $k \in \mathbb{N}$. We compute the location of the kth zero, z_k:

$$(-z_k + 1)\left(\frac{\pi}{2} + k\pi\right) = 1 + z_k$$

$$-z_k\frac{\pi}{2} - zk\pi + \frac{\pi}{2} + k\pi = 1 + z_k$$

$$\frac{\pi}{2} + k\pi - 1 = z_k + z_k\frac{\pi}{2} + z_k k\pi = z_k(1 + \frac{\pi}{2} + k\pi)$$

$$z_k = \frac{k\pi + \frac{\pi}{2} - 1}{k\pi + \frac{\pi}{2} + 1}$$

Comment. Notice that $\lim_{k\to\infty} z_k = 1$. This explains how $f(z)$ can have infinitely many zeroes converging to a limit point and yet not vanish, despite the identity theorem: the limit point isn't in the region that the function is holomorphic on.

2. *Proposition.* There does not exist a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that for all $n \in \mathbb{N}^+\backslash\{1\}$, $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$.

Comment. If we knew that $f(z) = f(-z)$ in general (not just for $\frac{1}{n}$, $n \in \mathbb{N}^+\backslash\{1\}$), then f would be an even function, which doesn't seem like it would play well with the power of 3 (which is odd) in the denominator of the output: *e.g.*, f can't be z^3 because while $\left(\frac{1}{n}\right)^3 = \frac{1}{n^3}$, $\left(-\frac{1}{n}\right)^3 = -\frac{1}{n^3} \neq \frac{1}{n^3}$. But this isn't yet a proof: we don't know that f is even, only that it behaves like an even function on $\frac{1}{n}$ ... but that's a convegent sequence, so we can probably invoke the identity theorem!

Proof. $x_n := \{\frac{1}{n}\}_{n=2}^{\infty}$ is a convergent sequence with a limit in $\mathbb{D}$. $g(x) := x^3$ takes on the values $\{\frac{1}{n^3}\}$ on x_n. By the identity theorem, the values of a function on the sequence uniquely determine the function, so if the proposition were true, we would have $f = g$. But $g(-\frac{1}{n}) = -\frac{1}{n^3} \neq \frac{1}{n^3} = g(\frac{1}{n})$, so $f \neq g$. Contradiction!

3. Let $f(z)$ be the monic polynomial

$$f(z) := z^n + \sum_{k=0}^{n-1} a_k z^k$$

Proposition. There exists z_0 such that $|z_0| = 1$ and $|f(z_0)| \geq 1$.

a. *Proof* (Cauchy). Based on a math.SE question by Don Fanucci (*https://math.stackexchange.com/q/2777129*) and an answer by Daniel (*https://math.stackexchange.com/a/276347*) on similar problems, I think the idea here is going to be to relate the Cauchy integral formula to the average value of a function on the unit circle $\frac{1}{2\pi}\int_0^{2\pi} f(\exp(it))\,dt$. If the average value on the $|z| = 1$ circle is 1, then there must exist a z_0 with $|z_0| = 1$ and $|f(z_0)| \geq 1$.

From the power rule, $f^{(n)}(z) = n!$. But by Cauchy's integral formula, we have

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_{|z|=1} \frac{z^n + \sum_{k=0}^{n-1} a_k z^k}{z^{n+1}} dz$$

Equating the two, we get

$$\cancel{n!} = \frac{\cancel{n!}}{2\pi i}\oint_{|z|=1} \frac{z^n + \sum_{k=0}^{n-1} a_k z^k}{z^{n+1}} dz$$

And then we reformulate in terms of the parametrized path $C = \exp(it)$ for $t \in [0, 2\pi]$. (With the change-of-variables factor $dz = C'(t)\,dt = i\exp(it)\,dt$.)

$$1 = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{\exp(it)^n + \sum_{k=0}^{n-1} a_k \exp(it)^k}{\exp(it)^{n\cancel{+1}}} \underbrace{\cancel{i}\cancel{\exp(it)}\,dt}_{dz}$$

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp(it)^n + \sum_{k=0}^{n-1} a_k \exp(it)^k}{\exp(it)^n} dt$$

$$1 = \frac{1}{2\pi}\int_0^{2\pi} 1 + \sum_{k=0}^{n-1} a_k \exp(it)^{k-n}\,dt$$

...?

b. *Proof* (Rouché). With regrets, I'm confused about how to use Rouché's theorem, which states that if $|f(z)| > |g(z)|$ on a circle, then f and $f + g$ have the same number of zeros inside the circle. How does the number of zeros give me information about $|f(z_0)| \geq 1$?

c. "There exists z_0 with $|z_0| = 1$ and $|f(z_0)| \geq |a_0|$"?

4. *Theorem.* A meromorphic function f is injective on $\mathbb{C}_\infty$ iff f is a Möbius transformation, *i.e.*, $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$.

Proof. ($\Leftarrow$) We show that if $\frac{az+b}{cz+d} = \frac{aw+b}{cw+d}$ and $ad - bc \neq 0$, then $z = w$.

$$(az + b)(cw + d) = (aw + b)(cz + d)$$

$$\cancel{acwz} + bcw + adz \cancel{+ bd} = \cancel{acwz} + bcz + adw \cancel{+ bd}$$

$$adz - bcz = adw - bcw$$

$$\cancel{(ad - bc)}z = \cancel{(ad - bc)}w$$

($\Rightarrow$) We know that meromorphic functions $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ are rational functions $\frac{P(z)}{Q(z)} = \frac{\sum_{j=0}^m a_j z^j}{\sum_{k=0}^m b_k z^k}$. We need to show that if f is injective, then $m, n \leq 1$—or to take the contrapositive, that if $m > 1$ or $n > 1$, then f is not injective, *i.e.*, that there exist $z \neq w$ such that $f(z) = f(w)$.

Suppose $m > 1$, which is to say that $\deg(P(z)) > 1$. By the fundamental theorem of algebra, $P(z)$ has $\deg(P(z))$ zeros, which are also zeros of f. If any two of those zeros z_1 and z_2 are distinct, then we have $z_1 \neq z_2$ and $f(z_1) = f(z_2) = 0$, and f is not injective.

Suppose $n > 1$, which is to say that $\deg(Q(z)) > 1$, the same argument goes through with loss of generality but with poles instead of zeros: by the fundamental theorem of algebra, $Q(z)$ has $\deg(Q(z))$ zeros—which are poles of f. If any two of those poles w_1 and w_2 are distinct, then we have $w_1 \neq w_2$ and $f(w_1) = f(w_2) = \infty$, and f is not injective.

Thus, it only remains to deal with the case where the zeros or poles of f are not distinct, where $f := \frac{(z-r)^m}{(z-s)^n}$ for root r, pole s, and $m \geq 2$ or $n \geq 2$. But that's not injective either, because any complex number has m mth roots and n nth roots. That is, if $f(z_0) = \frac{(z-a)^m}{(z-a)^n} = \frac{C}{D}$, then we could find a different z_1 by taking some $\frac{C^{\frac{1}{m}}+a}{D^{\frac{1}{n}}+a} \neq z_0$ (which has to exist, because there are $(n-1)(m-1)$ other choices for the fractional powers).

5. Let f be entire with $f(z) = f(z+1)$ and $|f(z)| \leq \exp|z|$, and $g(z) := \frac{f(z)-f(0)}{\sin(\pi z)}$.

Proposition. For $n \in \mathbb{Z}$, $g(n)$ is a removable singularity.

Proof. For $n \in \mathbb{Z}$, we know that $g(n)$ is a singularity because of the denominator $\sin(\pi n) = 0$. According to one form of Riemann's theorem on removable singularities, the singularity is removable if $\lim_{z\to n}(z-n)g(z) = \lim_{z\to n}(z-n)\frac{f(z)-f(0)}{\sin(\pi z)} = 0$. In the vicinity of n, we can use the local linear approximation $\sin \pi z \approx \pi(z-n)(-1)^n$ and the fact that $(-1)^{-n} = (-1)^n$ and the fact that $f(n) = f(0)$ (because $f(z) = f(z+1)$ implies that $f(n) = f(n-1) = f(n-2) = ... = f(0)$) to get:

$$\lim_{z\to n} \cancel{(z-n)}\frac{f(z)-f(0)}{\pi\cancel{(z-n)}(-1)^n} = \frac{1}{\pi}(-1)^n\cancelto{0}{(f(n)-f(0))} = 0$$

which is *quod erat demonstrandum.*

Proposition. g is bounded on $0 \leq \Re(z) < 1$.

Proof. We already know that f is bounded (by $\exp|z|$), so the only place where $g(z) = \frac{f(z)-f(0)}{\sin(\pi z)}$ could unboundedly "shoot off to infinity" is when the denominator is zero at $z = 1$. But we established in the previous proposition that those are removable singularities. Again by Reimann's theorem on removable singularities, g is bounded in a neighborhood of 1.

Proposition. f is constant.

Proof. The fact that $f(z) = f(z+1)$ implies that for any fixed y, $h_1(x) := f(x+iy)$ is periodic, and is therefore bounded by the extreme value theorem. (The function can't shoot off to infinity in between periods and still be continuous, but it has to be continuous because f is entire.) But for any fixed x, $h_2(y) := f(x+iy)$ is bounded because $|f(x+iy)| \leq \exp|x+iy| \leq \exp(|x|+|iy|) = \exp|x|\exp|iy| \leq \exp|x|$. It follows that f is bounded, and therefore constant by Liouville's theorem.

6. My initial thoughts: $f(z) = \frac{1}{z^{2k}(\exp z - 1)}$ has a pole of order $2k+1$ at $z = 0$ (because that's "how many times" the denominator gets zeroed out between $0^{2k} = 0$ and $\exp 0 - 1 = 0$).

We begin by calculating the residue as

$$\text{Res}(f, 0) = \frac{1}{(2k)!}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\left(\frac{z^{\cancel{2k}+1}}{\cancel{z^{2k}}(e^z-1)}\right) = \frac{1}{(2k)!}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\left(\frac{z}{\exp z - 1}\right)$$

We know that $\frac{z}{e^z-1}$ has the power series $\sum_{k=0}^{\infty}\frac{B_k}{k!}z^k$, which is to say that B_k is the kth derivative of $\frac{z}{\exp z - 1}$ at 0. Thus

$$\frac{1}{(2k)!}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\left(\frac{z}{\exp z - 1}\right) = \frac{B_{2k}}{(2k)!}$$

We are instructed to consider this residue in relation to a square contour of integration $\gamma_m := \pm(2m+1)\pi \pm (2m+1)\pi i$, and to deduce the value of $\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{n^{2k}}$, but to achieve this, we must grapple with the yet unexplained mystery of what this integral $\oint_{\gamma_m}\frac{1}{z^{2k}(e^z-1)}$ has to do with the zeta function ...

Sources online refer to George B. Arfken's *Mathematical Methods for Physicists* as a resource on this, and indeed, Afkin's example 7.2.6 concerns itself with residue methods for deriving the relationship between the Bernoulli numbers and the zeta function.

Arfken casts the $\left(\frac{z}{\exp z - 1}\right)^{(n)} = B_n$ relationship in terms of the Cauchy integral formula: $B_n = \frac{n!}{2\pi i}\oint\frac{z}{(\exp - 1)} \cdot \frac{1}{z^{n+1}}dz$.

There's some notational infelicities to navigate here, where Arfken is talking about B_n, but our problem statement is about B_{2n}, and I think we also haven't been totally consistent about the roles of the integers n and k. To clarify: I'm saying that $B_{2n} = \frac{(2n)!}{2\pi i} \oint \frac{1}{(\exp -1)} \cdot \frac{1}{z^{2n}} dz = \frac{(2n)!}{2\pi i} \oint f\, dz$.

And it would seem that I overlooked something critical about the pole situation above: in addition to the pole at 0 of order $2n+1$ for B_{2n}, we also have an additional countably many poles at $2\pi ki$ for $k \in \mathbb{Z}\backslash\{0\}$. That's going to be where the summation in the zeta function comes from.

$$\operatorname{Res}\left(\frac{z}{(\exp z - 1)} \cdot \frac{1}{z^{n+1}}, 2\pi ki\right) = \lim_{z\to 2\pi ki} (z - 2\pi ki) \frac{\cancel{z}}{(\exp z - 1)} \cdot \frac{1}{z^{n\cancel{+1}}} \underbrace{=}_{\text{L'Hôpital}} \lim_{z\to 2\pi ki} \frac{1}{\exp z} \cdot \frac{1}{z^n} = \frac{\overset{1}{\nearrow}1}{\cancel{\exp 2\pi ki}} \cdot \frac{1}{(2\pi ki)^n}$$

But we actually care about the even Bernoulli numbers B_{2n}. Arfken notes that the poles at $\pm 2\pi ki$ cancel for odd Bernoulli numbers, but add for even Bernoulli numbers. That's because $\frac{1}{(2\pi ki)^{2n}} = \frac{1}{(2\pi)^{2n} k^{2n} i^{2n}}$, where the even power of k means that $\operatorname{Res}(f, 2\pi ki) = \operatorname{Res}(f, -2\pi ki)$.

Also note that $i^{2n} = (-1)^n$. (Even powers of i, i.e., i^2, i^4, i^6, *&c.* alternate between -1 and 1.)

It seems like it should be possible to put all the pieces together now ... except that I'm still confused about what contour we should be using (which is kind of an important part of the problem!). The question prompt says to use the square γ_m, but Arfken is using a keyhole in which the poles at 0 is encircled with opposite orientation to the poles at $2\pi ki$ for $k \in \mathbb{Z}\backslash\{0\}$. These don't seem equivalent? Nevertheless we can calculate

$$B_{2n} = \frac{(2n)!}{2\pi i} \oint f\, dz$$

$$B_{2n} = \frac{(2n)!}{\cancel{2\pi i}} \cdot \cancel{2\pi i} \cdot 2 \lim_{m\to\infty} \sum_{k=1}^{m} \operatorname{Res}(f, 2\pi ik)$$

$$B_{2n} = 2(2n)! \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{(2\pi ki)^{2n}}$$

$$B_{2n} = \frac{2(2n)!}{(2\pi)^{2n}(-1)^n} \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^{2n}}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(2\pi)^{2n}(-1)^n B_{2n}}{2(2n)!}$$

which is *quod erat demonstrandum?*!