

Final

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22 May 2025

Abstract

“Take-home final” for Prof. Lai’s “Theory of Functions of a Complex Variable.”

1. *Proposition.* If an entire function satisfies $f(x+1) = f(x)$ for real x , then $f(z+1) = f(z)$ for all complex z .

Discussion. Sine and cosine are still periodic “in the real direction” even off the real axis (i.e., $\sin a + bi = \sin a + 2\pi + bi$, $\cos a + bi = \cos a + 2\pi + bi$ for nonzero b), which is suggestive that this might be true for non-constant entire functions in general. One idea for trying to construct a counterexample would be to take $f(z) := \sin z + g(z)$ for some entire g such that $g(z)$ is zero along the real axis (so that g will just be sine along the real axis and therefore remain periodic there), but nonperiodic for nonzero imaginary parts—but that doesn’t work because the only holomorphic function that’s zero on the real line is zero (by the identity theorem). (And the zero function is itself periodic, anyway.) The fact that this counterexample attempt fails is also suggestive, but we still need a proof. Can we do something with translating the real line to get a contradiction? (Translations of \mathbb{C} are holomorphic, and the sum of holomorphic functions is holomorphic.) But that doesn’t really seem to work—if we suppose for a contradiction that there exists a holomorphic function such that $f(x+1) = f(x)$ for $x \in \mathbb{R}$, but there exists $z_0 \in \mathbb{C}$ such that $f(z_0+1) \neq f(z_0)$, and then translate that, we just get a function that’s periodic on some horizontal line that’s not the real axis; it doesn’t show why it’s not possible for a holomorphic function to be periodic along one horizontal line but not another ...

Conducting a Google search for the query *holomorphic periodic only along the real line*, I come across a Math.StackExchange question asking about this exact problem.¹ The top answer says: “Hint #1: Examine the function $f(z) = g(z+1) - g(z)$ ” (with a mouseover hint #2 that I haven’t looked at), and another answer hints to use the identity theorem. OK, I see it.

Proof. Suppose for a contradiction that there exists a holomorphic function such that $f(x+1) = f(x)$ for $x \in \mathbb{R}$, but there exists $z_0 \in \mathbb{C}$ such that $f(z_0+1) \neq f(z_0)$. Consider $g(z) := f(z+1) - f(z)$. Then for $x \in \mathbb{R}$, $g(x) = 0$. But any point in \mathbb{R} is an accumulation point, so by the identity theorem, we must have $g(z) = 0$ for all $z \in \mathbb{C}$. But we also purportedly know that $g(z_0) = f(z_0+1) - f(z_0) \neq 0$. Contradiction!

2. *Proposition.* If $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic and there exist distinct points z_1 and z_2 such that $f(z_1) = z_1 = f(z_2)$, then $|f'(z_1)| < 1$.

Proof. Let $g := \frac{z_1 - z}{1 - \bar{z}_1 z}$. By the familiar properties of Blaschke factors, $g(z_1) = 0$, $g(0) = z_1$, and $g = g^{-1}$. We consider that $g \circ f \circ g(0) = g \circ f(z_1) = g(z_1) = 0$. As a mapping from the disk to itself that fixes 0, the Schwarz lemma applies to $g \circ f \circ g$. That lets us conclude that $|(g \circ f \circ g)'(0)| \leq 1$. In fact, because $f(z_2) = z_1$, f cannot be a rotation and the inequality is strict: $|(g \circ f \circ g)'(0)| < 1$.

Let us consider the nature of that derivative. By the chain rule, $(g \circ f \circ g)'(0) = \frac{d}{dz} g(f(g(0))) = g'(f(g(0))) \cdot f'(g(0)) \cdot g'(0)$. But because $f(z_1) = z_1$ and $g(0) = z_1$, we actually have $|g'(z_1) \cdot f'(z_1) \cdot g'(0)| < 1$.

But another, somewhat less familiar, property of Blaschke factors has to do with the modulus of their derivative at 0 and the point-being-mapped-to.² In our notation here: $|g'(0)| = 1 - |z_1|^2$ and $|g'(z_1)| = \frac{1}{1 - |z_1|^2}$. So we have

$$|g'(z_1) \cdot f'(z_1) \cdot g'(0)| = |g'(z_1)| \cdot |f'(z_1)| \cdot |g'(0)| = \frac{1}{1 - |z_1|^2} \cdot |f'(z_1)| \cdot \frac{1 - |z_1|^2}{1 - |z_1|^2} = |f'(z_1)| < 1$$

which is *quod erat demonstrandum!!*

3.a. If f is a non-constant entire function, then $a_k := \max_{2^k \leq |z| \leq 2^{k+1}} |f(z)|$ is strictly increasing.

¹<https://math.stackexchange.com/questions/3302250/entire-function-periodic-on-real-line-implies-periodic-on-mathbb>

²I found these formulas in David C. Ulrich’s *Complex Made Simple*.

Proof. Let $\overline{\Omega}_k = \{z : |z| \leq 2^{k+1}\}$. The maximum modulus principle implies that a non-constant holomorphic function on a compact set $\overline{\Omega}$ attains its maximum on the boundary $\partial\overline{\Omega}$. Therefore $\max_{2^k \leq |z| \leq 2^{k+1}} |f(z)| = \max_{\overline{\Omega}_k} |f(z)|$ (because the maximum cannot occur on z for which $|z| \leq 2^k$) and in particular $|\arg \max_{\overline{\Omega}_k} |f(z)| = 2^{k+1}$.

We need to show that for all k , $a_{k+1} > a_k$. Let k be arbitrary. Now a_k is the maximum on $\overline{\Omega}_k$, and a_{k+1} is the maximum on $\overline{\Omega}_{k+1}$. But $\overline{\Omega}_k \subset \overline{\Omega}_{k+1}$, so a_{k+1} being the maximum on $\overline{\Omega}_{k+1}$ implies that $a_{k+1} > a_k$. (Another way to put it: if $a_k > a_{k+1}$, then a_k would be an internal (not on the boundary) maximum of $\overline{\Omega}_{k+1}$, which would contradict the maximum modulus principle.)

b. Proposition. If f is entire and $|f(z^2)| \leq |f(z)|$ for all $|z| \geq 1$, then f is constant.

Proof. If $|z| > 1$, then $|z^2| > |z|$. If $|f(z^2)| \leq |f(z)|$, that implies that the maximum of f on (let's say, for concreteness) $\{z : |z| \leq 4\}$ does *not* occur on $|z| = 4$, because $|f(z)|$ takes greater values on $|z| = 2$. That contradicts the maximum modulus principle for non-constant holomorphic functions, so f must be constant.

4. Proposition. There exists a holomorphic function on $\{z : |z| > 4\}$ such that its derivative is equal to $\frac{z}{(z-1)(z-2)^2}$, and there does not exist a holomorphic function on $\{z : |z| > 4\}$ such that its derivative is equal to $\frac{z^2}{(z-1)(z-2)^2}$.

Discussion. The mention of the derivative and the circle $|z| = 4$ makes me suspect that this might involve the argument principle, which states that $\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i(Z - P)$ where Z is the number of zeros and P the number of poles inside C , but I'm not seeing how to apply that. (The intuition was that information about the derivative could tell us about poles, which would tell us that the function can't be holomorphic on a region including the pole.) WolframAlpha (saving me the trouble of doing the partial fraction decomposition myself) says that $\int \frac{z}{(z-1)(z-2)^2} = -\frac{2}{z-2} + \log(1-z) + \log(2-z)$ and $\int \frac{z^2}{(z-1)(z-2)^2} = -\frac{4}{z-2} + \log(z-1)$, which both have singularities at 1 and 2. But then I'm getting $\frac{f'}{f} = \frac{\frac{z}{(z-1)(z-2)^2}}{-\frac{2}{z-2} + \log(1-z) + \log(2-z)} = \frac{z}{(-\frac{2}{z-2} + \log(1-z) + \log(2-z))(z-1)(z-2)^2}$, which does not appear to be readily integrable (and WolframAlpha doesn't think so, either)? So probably the argument principle is not the right tack here.

Conducting a Google search for *complex analysis when does a primitive exist* turns up a Math.StackExchange question asking about a similar problem,³ where the discussion hinges on a claim that $p \log(z-1) + q \log(z-2) + r \log(z-3)$ is well-defined where $p + q + r = 0$, but this assertion isn't obvious to me and the commenters don't explain it at all ... OK, on further consideration and studying the discussion, I think the idea here is that a closed loop in $\{z : |z| > 4\}$ that encloses the "hole" $\{z : |z| \leq 4\}$ still needs to be zero in order for a primitive to exist; we need the poles to cancel out.

Proof. We perform partial fraction decomposition. If $\frac{z}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$, then $z = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$. If $z := 1$, then $1 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$ implies $A = 1$. If $z := 2$, then $2 = A(2-2)^2 + B(2-1)(2-2) + C(2-1)$ implies $C = 2$. So we've deduced $z = (z-2)^2 + B(z-1)(z-2) + 2(z-1)$. Then if $z := 3$, we have $3 = (3-2)^2 + B(3-1)(3-2) + 2(3-1)$ implies $3 = 1 + B(2)(1) + 2(2)$ implies $2 = 2B + 4$ implies $-2 = 2B$ implies $B = -1$. So we have $\frac{z}{(z-1)(z-2)^2} = \frac{1}{z-1} + \frac{-1}{z-2} + \frac{2}{(z-2)^2}$.

A primitive will exist on $\{z : |z| > 4\}$ if every closed loop integrates to zero—including loops that circle the poles in $\{z : |z| \leq 4\}$. So we compute the residues: $\text{Res}_{\frac{1}{z-1}} = \lim_{z \rightarrow 1} (z-1) \frac{1}{z-1} = 1$, $\text{Res}_{\frac{-1}{z-2}} = \lim_{z \rightarrow 2} (z-2) \frac{-1}{z-2} = -1$, $\text{Res}_{\frac{2}{(z-2)^2}} = \lim_{z \rightarrow 2} \frac{d}{dz} (z-2)^2 \frac{2}{(z-2)^2} = 0$. Because the residues sum to zero, $\oint_\gamma \frac{z}{(z-1)(z-2)^2} = 0$ for γ in $\{z : |z| > 4\}$ and a primitive exists.

The partial fraction decomposition of $\frac{z^2}{(z-1)(z-2)^2}$ is $\frac{4}{(z-2)^2} + \frac{1}{z-1}$. (Computation delegated to WolframAlpha; I demonstrate above that I can do it by hand if I need to.) But here $\text{Res}_{\frac{4}{(z-2)^2}} = 0$ but $\text{Res}_{\frac{1}{z-1}} = 1$, so the residues don't cancel, the loop integrals don't vanish, and the primitive doesn't exist.

5. a. Proposition. $\hat{f}(z) := \int_0^1 f(x) \exp(-2\pi izx) dx$ (for f such that $\int_0^1 |f(x)| dx < \infty$) is an entire function.

Discussion. My first thought was that maybe evaluating the integral with respect to x would help us show why this is an entire function with respect to z , so I tried integrating by parts, and after some calculation (omitted) arrived at $\int_0^1 f(x) \exp(-2\pi izx) dx = \frac{F(1) \exp(-2\pi iz) - F(0)}{1 - 2\pi iz}$ (where $F'(x) = f(x)$), which isn't entire because it has a pole at $\frac{1}{2\pi i}$ —so presumably I made a mistake somewhere. Then reading about the Fourier transform online led me to the Paley–Weiner theorem, and that Ch. 4 of Stein and Shkarchich is specifically about this. The most relevant result (Theorem 3.1) in that chapter appeals to Theorem 5.4 in Chapter 2, whose proof I now adapt to this circumstance.

³<https://math.stackexchange.com/questions/57368/finding-a-primitive-explicitly>

Proof. Consider $\hat{f}_n(z) = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \exp(-2\pi iz \frac{k}{n})$. (Motivation: \hat{f}_n is a Riemann sum; the big idea here is that we're going to have $\lim_{n \rightarrow \infty} \hat{f}_n = \hat{f}$.) Note that for any given n , \hat{f}_n is holomorphic (as the exponential $\exp(-2\pi iz \frac{k}{n})$ is holomorphic, and multiplying it by the constant $f\left(\frac{k}{n}\right)$ doesn't change that).

Consider an arbitrary compact disk $D \subset \mathbb{C}$. We aim to show that \hat{f}_n converges uniformly to \hat{f} on D . Fix ε .

At the expense of making this proof slightly less general than it could be, let's suppose that f is continuous, so that $f(x) \exp(-2\pi izx)$ is continuous (and thus uniformly continuous on a compact disk). Then there exists a δ such that when $|x - \frac{k}{n}| < \delta$, then $\sup_{z \in D} |f\left(\frac{k}{n}\right) \exp(-2\pi iz \frac{k}{n}) - f(x) \exp(-2\pi izx)| < \varepsilon$. Then we have

$$|\hat{f}_n(z) - \hat{f}(z)| = \left| \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \exp\left(-2\pi iz \frac{k}{n}\right) - \int_0^1 f(x) \exp(-2\pi izx) dx \right|$$

which ("slicing up" the integral to make it compatible with the sum)

$$\begin{aligned} &= \left| \sum_{k=1}^n f\left(\frac{k}{n}\right) \exp\left(-2\pi iz \frac{k}{n}\right) - \int_{(k-1)/n}^{k/n} f(x) \exp(-2\pi izx) dx \right| \\ &\leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f\left(\frac{k}{n}\right) \exp\left(-2\pi iz \frac{k}{n}\right) - f(x) \exp(-2\pi izx) \right| dx < \sum_{k=1}^n \frac{\varepsilon}{n} = \varepsilon \end{aligned}$$

Thus, \hat{f}_n is a sequence of entire functions that converges uniformly to \hat{f} on every closed disk in \mathbb{C} ; thus, by Theorem 5.2 of Stein and Shakarchi's Chapter 2, \hat{f} itself is entire.

b. Proposition. $\Phi = \left\{ \hat{f} : \int_0^1 |f(x)| dx \leq M \right\}$ is an equicontinuous family.

Discussion. We already know that the exponential function is continuous, so we can adapt techniques from the proof that $\exp(x)$ is continuous⁴ to show that this integral transform is continuous independently of the choice of f from the family.

Proof. We want to show that for all ε and for all $z_0 \in \mathbb{C}$ and for all $f \in \Phi$, there exists a δ such that if $|z - z_0| < \delta$, then $\left| \int_0^1 f(x) \exp(-2\pi izx) dx - \int_0^1 f(x) \exp(-2\pi iz_0x) dx \right| < \varepsilon$.

Fix ε . We reason as follows:

$$\begin{aligned} &\left| \int_0^1 f(x) \exp(-2\pi izx) dx - \int_0^1 f(x) \exp(-2\pi iz_0x) dx \right| = \left| \int_0^1 f(x) (\exp(-2\pi izx) - \exp(-2\pi iz_0x)) dx \right| \\ &\left| \int_0^1 f(x) \exp(-2\pi iz_0x) (\exp(2\pi ix(z_0 - z)) - 1) dx \right| \leq \int_0^1 |f(x)| \exp(-2\pi iz_0x) |\exp(2\pi ix(z_0 - z)) - 1| dx \\ &= \int_0^1 |f(x)| \exp(-2\pi iz_0x) |\exp(2\pi ix(z_0 - z)) - \exp(z_0 - z_0)| dx \end{aligned}$$

But we already know that the exponential function is continuous, which means there exists a δ such that if $|z_0 - z| < \delta$, then $|\exp(2\pi ix(z_0 - z)) - \exp(z_0 - z_0)|$ is arbitrarily small. In particular, let $E := \frac{-2\pi iz_0}{M(\exp(-2\pi iz_0) - 1)} \varepsilon$, and pick δ such that $|\exp(2\pi ix(z_0 - z)) - \exp(z_0 - z_0)| < E$. Then we have

$$\begin{aligned} &\int_0^1 |f(x)| \exp(-2\pi iz_0x) |\exp(2\pi ix(z_0 - z)) - \exp(z_0 - z_0)| dx \leq EM \int_0^1 \exp(-2\pi iz_0x) dx \\ &= \frac{EM}{-2\pi iz_0} (\exp(-2\pi iz_0) - 1) = \frac{M}{-2\pi iz_0} \frac{\exp(-2\pi iz_0) - 1}{\exp(-2\pi iz_0) - 1} \frac{-2\pi iz_0}{M(\exp(-2\pi iz_0) - 1)} \varepsilon = \varepsilon \end{aligned}$$

⁴I consulted "Wrath of Math"'s YouTube video on the continuity of the exponential for reference: <https://www.youtube.com/watch?v=CPW08RsIBy8>

6. Proposition. If f_k is a sequence of holomorphic functions defined on an open set containing $\overline{\mathbb{D}}$ such that f_n converge uniformly on $\partial\mathbb{D}$, then $f := \lim_{n \rightarrow \infty} f_n$ is holomorphic on \mathbb{D} .

Proof.⁵ Let f_k be a sequence of holomorphic functions converging uniformly on $\partial\mathbb{D}$. By the Cauchy criterion for uniform convergence, that means that for all $z \in \partial\mathbb{D}$ and for all ε , there exists N such that if $m, n \geq N$, then $|f_m(z) - f_n(z)| < \varepsilon$.

By the maximum modulus principle, the point where $|f_m(z) - f_n(z)|$ attains its maximum over $\overline{\mathbb{D}}$ is on the boundary $\partial\mathbb{D}$; call this point (where the maximum is attained) z_0 . Then for any $z \in \mathbb{D}$ with $|z| < |z_0|$, we have $|f_m(z) - f_n(z)| < |f_m(z_0) - f_n(z_0)| < \varepsilon$, so the sequence $\{f_k\}$ also converges uniformly (with the same N) on \mathbb{D} , not just $\partial\mathbb{D}$.

Theorem 5.2 of Chapter 3 of Stein and Shakarchi states that if a sequence of holomorphic functions converges uniformly in every compact subset of a region Ω , then the limit is also holomorphic in Ω . Here we can take $\Omega := \mathbb{D}$ (since $\{f_k\}$ converges uniformly in all of \mathbb{D} , it certainly also does so in any compact subset thereof) and conclude that f is holomorphic.

7. Proposition. Let

$$\mathcal{F} := \left\{ f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ holomorphic} \ \& \ \sup_{r < 1} \int_0^{2\pi} |f(r \exp i\theta)| \, d\theta \leq 1 \right\}$$

Then \mathcal{F} is a normal family.

Discussion. Theorem 3.3 in Chapter 8 of Stein and Shakarchi states that a family of holomorphic functions that is uniformly bounded on compact subsets of its domain is a normal family: *i.e.*, if there exists B such that $|f(z)| \leq B$ for all $z \in K \subset \mathbb{D}$ (K being compact) and $f \in \mathcal{F}$. Then it's tempting to say that (properly) integrable functions are bounded, so the fact that $\int_0^{2\pi} |f(r \exp i\theta)| \, d\theta$ is integrable at all implies that $|f(r \exp i\theta)|$ is bounded for all $r < 1$. But that doesn't quite work as stated. ("Bounded for any particular r " doesn't mean "bounded" if the particular bounds can increase with r .) You might hope that the integral itself being bounded by 1 would suffice, but I'm not even confident of that, because a pathological mind could imagine that the integrals for increasing values of r have an increasingly high "spike" on a set of increasingly tiny measure such that the integrals stay bounded but $|f(r \exp i\theta)|$ doesn't. I can't even dismiss this scenario as intuitively implausible: a function with a pole at z_0 with $|z_0| = 1$ would seem to have such a spike and is holomorphic on the *open* unit disk (not including the boundary $\partial\mathbb{D}$ on which the pole lies).

The hint says to review the proof of Montel's theorem. Could the diagonalization trick somehow be useful here?! I'm not seeing it.

Performing a Google search for *prove that bounded integral implies normal family* reveals a couple of Math.StackExchange discussions of (essentially) this problem; I'm reading a couple of answers^{6,7} making use of Cauchy's integral formula, which I hopefully understand well enough to write in my own words?

Proof. Consider an arbitrary compact $K \subset \mathbb{D}$. For $z \in K$ and $f \in \mathcal{F}$ and $|z| < r < 1$, we have by Cauchy's integral formula:

$$|f(z)| = \left| \frac{1}{2\pi i} \oint \frac{f(Z)}{Z - z} \, dZ \right| = \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(r \exp i\theta)|}{|r \exp i\theta - z|} \, d\theta = \frac{1}{2\pi |r - |z||} \underbrace{\int_0^{2\pi} |f(r \exp i\theta)| \, d\theta}_{\leq 1} \leq \frac{1}{2\pi |r - |z||}$$

But by bounding an arbitrary function from \mathcal{F} in a compact neighborhood of an arbitrary point, we show that the function is uniformly bounded on compact subsets and therefore \mathcal{F} is a normal family by Theorem 3.3 in Chapter 8 of Stein and Shakarchi—*quod erat demonstrandum?*

⁵Thanks to a Math.StackExchange discussion of this question for pointing out the relevance of the maximum modulus principle: <https://math.stackexchange.com/questions/4745971/uniform-convergence-of-a-sequence-of-holomorphic-functions-on-the-unit-circle-an>

⁶<https://math.stackexchange.com/questions/4694840/show-that-mathcalf-is-a-normal-family>

⁷<https://math.stackexchange.com/questions/1611778/contour-integral-inequality-and-normal-families>