Assignment #7

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Abstract

Homework exercises for Prof. Lai's "Theory of Functions of a Complex Variable."

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1. *Proposition.* $\int_0^\infty \frac{1}{1+x^3}\ dx = \frac{2\pi}{3\sqrt{3}}$

Proof. $\frac{1}{1+z^3}$ has poles where $z^3+1=0$, thus, where $z^3=-1$. That would be $\exp(\frac{1}{3}\pi i)$, -1, and $\exp(\frac{5}{3}\pi i)$.

We can integrate along a wedge-shaped region: from 0 to R along the real axis, counterclockwise along a circular arc of angle $\frac{2\pi}{3}$, and back to the origin.[1]

The integral on the segment of the real line is $\int_0^R \frac{1}{1+r^3}\,dr$.

We can bound the integral on the arc using the estimation lemma, and see that it vanishes for large R: $|\int \frac{1}{1+x^3}| \leq \frac{1}{1+R^3}\cdot\frac{1}{2}\pi R = \frac{\pi R}{2(R^3+1)}$ but $\lim_{R\to\infty}\frac{\pi R}{2(R^3+1)} = 0$, because the cubic denominator dominates the linear numerator.

The integral from the arc to the origin is $\int_{R\exp(\frac{2\pi}{3}i)}^0 \frac{1}{1+z^3}dz = \int_R^0 \frac{1}{1+\left(t\exp(\frac{2}{3}\pi i)\right)^3}\underbrace{\exp(\frac{2\pi i}{3})dt}_{dz} = \int_R^0 \frac{1}{1+t^3\cdot\cancel{\exp(\frac{2\cdot 3}{3}\pi i)}}\exp(\frac{2\pi i}{3})dt = -\exp(\frac{2\pi i}{3})\int_0^R \frac{1}{1+t^3}dt$.

The only pole inside the loop is the one at $\exp(\frac{1}{3}\pi i)$. The residue there is $\lim_{z\to\exp(\frac{1}{3}\pi i)}(z-\exp(\frac{1}{3}\pi i))\frac{1}{1+x^3} = \lim_{z\to\exp(\frac{1}{3}\pi i)}\cancel{(z-\exp(\frac{1}{3}\pi i))}\frac{1}{\cancel{(z-\exp(\frac{1}{3}\pi i))}(z+1)(z-\exp(\frac{5}{3}\pi i))} = \frac{1}{(\exp(\frac{1}{3}\pi i)+1)(\exp(\frac{1}{3}\pi i)-\exp(\frac{5}{3}\pi i))}$.

Let $r := \exp(\frac{1}{3}\pi i)$. The denominator is $(r+1)(r-\overline{r}) = (r+1)\cdot 2i\Im(r) = \left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\cdot 2i\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}i + \frac{3}{2}i^2 = -\frac{3}{2}+\frac{3\sqrt{3}}{2}i$.

So we ultimately have $-\exp(\frac{2\pi i}{3})\int_0^R \frac{1}{1+t^3}dt + \int_0^R \frac{1}{1+t^3}dt = \frac{2\pi i}{-\frac{3}{2}+\frac{3\sqrt{3}}{2}i}$, which implies

$$\int_0^R \frac{1}{1+t^3}dt = \frac{2\pi i}{\left(1-\exp(\frac{2\pi i}{3})\right)\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)}$$

It was temping to "call it a day" there, but if we persist in simplying, we get: $\frac{2\pi i}{\left(1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)} = \frac{2\pi i}{\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)} = \frac{2\pi i}{-\frac{9}{4}+\frac{3\sqrt{3}}{4}i+\frac{9\sqrt{3}}{4}i\cancel{-\frac{9}{4}i^2}} = \frac{2\pi i}{\cancel{-\frac{9}{4}}+\frac{3\sqrt{3}}{4}i+\frac{9\sqrt{3}}{4}i\cancel{+\frac{9}{4}}}$, and thus:

$$\frac{2\pi i}{\frac{12\sqrt{3}}{4}i} = \frac{2\pi\cancel{i}}{3\sqrt{3}\cancel{i}} = \frac{2\pi}{3\sqrt{3}}$$

2 (Ch. 3 #3). *Proposition.*

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx = \pi\frac{\exp -a}{a}$$

[1] Thanks to Claude Sonnet 3.7 for pointing out the special properties of this contour, that it lets us algebraically manipulate $\int\frac{1}{1+t^3}dt$ (and for reminding me about the change-of-variables factor when we parametrize). I had originally tried to use a quarter-disk contour involving a segment of the imaginary axis, and got stuck when I didn't see how to evaluate $\int_{iR}^0 \frac{1}{1+z^3}\,dz$. (WolframAlpha computes the antiderivative as $\int\frac{1}{1+x^3}\,dx = \frac{1}{6}(-\log(x^2-x+1)+2\log(x+1)+2\sqrt{3}\arctan(\frac{2x-1}{\sqrt{3}}))+C$, which does not look tidy enough to work with.)

Proof. We consider that $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} = \Re\left(\int_{-\infty}^{\infty} \frac{\exp ix}{x^2+a^2}\, dx\right)$, where the right-hand integrand has a pole at ia.[2]

We integrate on a region bounded by the real axis from $[-R, R]$ and a semicircular arc in the upper-half plane. In the limit of large R, the integral over the arc vanishes (exponentially decaying numerator, quadratic denominator): $\lim_{|z|\to\infty} \frac{\exp iz}{z^2+a^2} = \lim_{|z|\to\infty} \frac{\exp i(r\cos\theta + ir\sin\theta)}{z^2+a^2} = \lim_{|z|\to\infty} \frac{\exp(ri\cos\theta - r\sin\theta)}{z^2+a^2} = \lim_{|z|\to\infty} \frac{\exp(-r\sin\theta)\exp(ri\cos\theta)}{z^2+a^2} = 0$. So the real integral will be equal to the loop integral, which is determined by the residue:

$$2\pi i \mathrm{Res}(\frac{\exp iz}{z^2+a^2}, ia) = 2\pi i \lim_{z\to ia} \cancel{(z-ia)} \frac{\exp iz}{\cancel{(z-ia)}(z+ia)} = \cancel{2}\pi\cancel{i} \frac{\exp -a}{\cancel{2i}a}$$

3 (Ch. 3 #6). *Proposition.* For $n \geq 1$,

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx = \frac{\prod_{j=1}^{n} 2n-1}{\prod_{j=1}^{n} 2n} \pi$$

Initial comment. Just looking at the integrand, one imagines this is going to go down similarly to $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$, but with the poles being of higher order?

Proof. We find the poles, which are of order $n+1$:

$$f(z) := \frac{1}{(1+z^2)^{n+1}} = \frac{1}{((z-i)(z+i))^{n+1}} = \frac{1}{(z-i)^{n+1}(z+i)^{n+1}}$$

If we integrate in a upper-half-plane semi-circular arc of radius R, the integral of the arc will go to zero as $R \to \infty$: $f(z)$ asymptotically behaves like $\frac{1}{z^{2n+2}}$ and the estimation lemma says that $|\int f(z)\, dz| \leq \frac{\pi R}{R^{2n+2}} = \frac{\pi}{R^{2n+1}}$, which goes to 0 as $R \to \infty$.

We calculate the residue:

$$\mathrm{Res}(f, i) = \frac{1}{n!} \lim_{z\to i} \frac{d^n}{dz^n} \left(\cancel{(z-i)^{n+1}} \frac{1}{\cancel{(z-i)^{n+1}}(z+i)^{n+1}} \right) = \frac{1}{n!} \lim_{z\to i} \frac{d^n}{dz^n} \left((z+i)^{-n-1}\right)$$

$$= \frac{1}{n!} \lim_{z\to i} (-1)^n \frac{(2n)!}{n!} \left((z+i)^{-n-1-n}\right) = (-1)^n \frac{(2n)!}{n!^2} (2i)^{-2n-1} = (-1)^n \frac{(2n)!}{n!^2} \cdot \frac{1}{2^{2n+1}} \cdot i^{-2n-1}$$

And apparently there's a "double" or skip factorial identity to apply here: $(2n)! = 2^n n! \prod_{k=1}^{n}(2k-1)$,[3] so we have

$$(-1)^n \frac{\cancel{2^n}\cancel{n!} \prod_{k=1}^{n}(2k-1)}{n!^{\cancel{2}} 2 \cdot 2^n \cdot \cancel{2^n}} \cdot i^{-2n-1} = (-1)^n \frac{\prod_{k=1}^{n}(2k-1)}{2 \cdot 2^n n!} \cdot i^{-2n-1} = (-1)^n \frac{\prod_{k=1}^{n}(2k-1)}{2 \cdot \prod_{k=1}^{n} 2k} \cdot i^{-2n-1}$$

And the value of the loop integral is $2\pi i$ times the residue, so that's

$$\cancel{2}\pi\cancel{(-1)^n} \frac{\prod_{k=1}^{n}(2k-1)}{\cancel{2}\cdot \prod_{k=1}^{n} 2k} \cdot \cancelto{\frac{1}{(-1)^n}}{i^{-2n}}$$

$$= \frac{\prod_{k=1}^{n}(2k-1)}{\prod_{k=1}^{n} 2k} \pi$$

4 (Ch. 3 #12). *Proposition.*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$$

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[2]Thanks to Claude Sonnet 3.7 for explaining this approach; I had tried computing the residue of $f(z) = \frac{\cos x}{x^2+a^2}$: $\mathrm{Res}(f, ia) = \lim_{z\to ia}(z-ia)\frac{\cos z}{z^2+a^2} = \lim_{z\to ia} \cancel{(z-ia)} \frac{\cos z}{\cancel{(z-ia)}(z+ia)} = \frac{\cos ia}{2ia}$. But $\cos ia = \frac{\exp(i^2 a)+\exp(-i^2 a)}{2} = \frac{\exp(-a)+\exp(a)}{2}$, so $\frac{\cos ia}{2ia} = \frac{\exp(-a)+\exp(a)}{4ia}$. Then $2\pi i$ times the residue is $\cancel{2}\pi\cancel{i}\frac{\exp(-a)+\exp(a)}{\cancelto{2}{4}\cancel{i}a} = \pi\frac{\exp(-a)+\exp(a)}{2a}$. But then, I'm given to understand, we can't use the semicircular contour technique because the arc doesn't vanish.

[3]Thanks to OpenAI o3-mini-high for pointing this out; I had never worked with double factorial before in my life.

Proof. $f(z) = \frac{\pi}{(u+z)^2}\frac{\cos \pi z}{\sin \pi z}$ for non-integer u has poles at integers (where $\sin \pi z$ is zero) and at $-u$ (where $(u+z)^2$ is zero).

The pole at $z := -u$ has degree two. Looking up that "$\frac{d}{dt} \cot t = -\csc^2 t$" (the present author not often having occasion to think of cotangent or cosecant) and recalling that sine is an odd function, we compute

$$\mathrm{Res}(f, -u) = \lim_{z \to -u} \frac{d}{dz} \cancel{(z+u)^2} \frac{\pi}{\cancel{(u+z)^2}} \frac{\cos \pi z}{\sin \pi z} = \lim_{z \to -u} \frac{-\pi}{\sin^2 \pi z}\pi = \frac{-\pi^2}{\sin^2(-\pi u)} = \frac{-\pi^2}{(-\sin \pi u)^2} = \frac{-\pi^2}{\sin^2 \pi u}$$

The integer poles are simple. We calculate

$$\mathrm{Res}(f, k) = \lim_{z \to k} (z-k) \frac{\pi}{(u+z)^2} \frac{\cos \pi z}{\sin \pi z}$$

Using the local linear approximations for the trig functions ($\sin \pi z \approx \pi(z-k)(-1)^k$ and $\cos \pi z = (-1)^k$) near k:[4]

$$\approx \cancel{(z-k)} \frac{\cancel{\pi}}{(u+z)^2} \cdot \frac{\cancel{(-1)^k}}{\cancel{\pi}\cancel{(z-k)}\cancel{(-1)^k}} = \frac{1}{(u+k)^2}$$

We integrate on a sequence of increasingly large square contours $|x| = |y| = k + \frac{1}{2}$. Then $\frac{\cos \pi z}{\sin \pi z}$ is bounded on these contours because $|\cos(x+iy)| = |\frac{\exp(i(x+iy)) + \exp(-i(x+iy))}{2}| = |\frac{\exp(-y+ix)+\exp(y-ix)}{2}|$ and similarly $|\sin \pi(x+iy)| = |\frac{\exp(-y+ix) - \exp(y-ix)}{2i}|$, so their ratio stays bounded. and the inverse-quadratic $\frac{1}{(u+z)^2}$ factor dominates for large N, such that the value of $f(z) \to 0$ for large N, such that the loop integral $\oint f(z)\, dz$ is zero. Then

$$\oint f(z)\, dz = 0 = 2\pi i \mathrm{Res}(f, -u) + 2\pi i \sum_{k=-\infty}^{\infty} \mathrm{Res}(f, k) = \frac{-\pi^2}{\sin^2 \pi u} + \frac{1}{(u+k)^2}$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{(u+k)^2} = \frac{\pi^2}{\sin^2 \pi u}$$

[4] Thanks to Claude Sonnet 3.7 for pointing out that this works; I had initally tried to iteratively apply L'Hôpital's rule and made a mess with no trustworthy conclusion. (Trying to apply the product rule still left me with terms with $\sin \pi k = 0$ in the denominator, as in $\pi \lim_{z \to k} \frac{-\pi \sin \pi z}{(u+z)^2 \sin \pi z} - \frac{2(u+z)\sin \pi z + \pi (u+z)^2 \cos \pi z}{((u+z)^2 \sin \pi z)^2}$; I'm still not entirely sure which step was incorrect.)