Assignment #6

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Abstract

Homework exercises for Prof. Lai's "Theory of Functions of a Complex Variable."

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1 (Ch. 2 Exercise 7). *Proposition.* Suppose $f : \overline{B_1}(0) \to \mathbb{C}$ is holomorphic. Then $2|f'(0)| \leq \sup_{z,w \in \overline{B_1}(0)} |f(z) - f(w)|$.

Proof.[1] By Cauchy's integral formula, $f'(z) = \frac{1}{2\pi i} \oint \frac{f(Z)}{(Z-z)^2} dZ$, so $f'(0) = \frac{1}{2\pi i} \oint \frac{f(Z)}{Z^2} dZ$

Let $g(z) := f(-z)$. Then $g'(z) = -f'(-z)$. Also, $g'(z) = \frac{1}{2\pi i} \oint \frac{g(Z)}{(Z-z)^2} dZ = \frac{1}{2\pi i} \oint \frac{f(-Z)}{(Z-z)^2} dZ$. So we also have $g'(0) = -f'(-0) = \frac{1}{2\pi i} \oint \frac{f(-Z)}{Z^2} dZ$.

Thus $2f'(0) = \frac{1}{2\pi i} \oint \frac{f(Z) - f(-Z)}{Z^2} dZ$. Applying the estimation lemma around a unit-circle contour, we get

$$2|f'(0)| \leq \left| \frac{1}{2\pi i} \right| \sup_{|Z|=1} \left| \frac{f(Z) - f(-Z)}{Z^2} \right| \cdot 2\pi = \sup_{|Z|=1} |f(Z) - f(-Z)| \leq \sup_{z,w \in \overline{B_1}(0)} |f(z) - f(w)|$$

Proposition. ... with equality if f is linear.

Proof.

$$\sup_{z,w \in \overline{B_1}(0)} |f(z) - f(w)| = \sup_{z,w \in \overline{B_1}(0)} |az \cancel{+b} - aw \cancel{-b}| = \sup_{z,w \in \overline{B_1}(0)} a|z - w| = 2a = 2|f'(0)|$$

2 (Ch. 2 Exercise 12a). *Proposition.* Suppose $\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Then there exists holomorphic $f := u(x, y) + iv(x, y)$, with v unique up to an additive constant.

Proof.[2] For a function $u(x, y)$, consider whether the function $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ might be holomorphic. The Cauchy–Riemann equations (with the "roles" $u := \frac{\partial u}{\partial x}$ and $v := \frac{\partial u}{\partial y}$) become $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial y^2}$ (which amounts to stating that u is harmonic) and $\frac{\partial u}{\partial x \partial y} = \frac{\partial u}{\partial y \partial x}$ (which amounts to the equality of mixed partials). So given that u is in fact harmonic and twice differentiable, those C–R equations are fulfilled, such that $\mathrm{Im}(\int \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})$ (with an arbitrary additive constant of integration) will the imaginary part of a holomorphic function whose real part is u.

3 (Ch. 2 Problem 1b diluted). *Proposition.* $f(z) = \sum_{j=1}^{\infty} 2^{-\alpha j} z^{2^j}$ defines a continuous function on the unit disk, but cannot be analytically continued outside of the unit disk.

Proof.[3] Inside the unit disk, $|2^{-\alpha j} z^{2^j}| < 2^{-\alpha j}$ (because for $|z| < 1$ and all m, $|z|^m < 1$). But $\sum_{j=1}^{\infty} 2^{-\alpha j}$ is a convergent geometric series (with ratio $\frac{1}{2^\alpha}$), so we can invoke the Weierstrass M-test with $M_j := 2^{-\alpha j}$ to conclude that $\sum_{j=1}^{\infty} 2^{-\alpha j} z^{2^j}$ is uniformly and absolutely convergent inside the unit disk. As a power series, it is continuous.

On the unit circle, we have $f(z) = \sum 2^{-\alpha n} \exp(2^n i\theta)$. Applying the de Moivre identity, those terms are $\underbrace{2^{-\alpha n} \cos(2^n \theta)}_{u} + i \underbrace{2^{-\alpha n} \sin(2^n \theta)}_{v}$. But u and v are known to be non-differentiable, so the (polar) Cauchy–Reimann equations $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ cannot be satisfied.

[1] Thanks to Ophir Horovitz for pointing out that the Stein and Shakarchi text had a hint which I had overlooked, and guidance on how to apply it, and Claude Sonnet 3.5 for further guidance on how to related the Cauchy integral formula integral to the desired result.

[2] Thanks to the "Harmonic Conjugate" *Wikipedia* article (*https://en.wikipedia.org/wiki/Harmonic_ conjugate*) for finally providing a clear explanation after much confusion.

[3] Thanks to OpenAI o3-mini-high for hints: pointing out that $2^{-\alpha j}$ is a bounding geometric series, and suggesting to look at a point on the circle and separate real and imaginary parts.

4 (Ch. 2 Problem 2 diluted). **a**. *Proposition.* The radius of convergence of $F(z) = \sum_{k=1}^{\infty} d(k)z^k$ (where $d(k)$ is the number of divisors of k) is 1.

Proof. We apply the root test:

$$\limsup_{k\to\infty} |d(k)|^{\frac{1}{k}}$$

Because $1 \le d(k) \le k$, we can apply the squeeze theorem[4]: $\limsup |1|^{\frac{1}{k}} \le \limsup_{k\to\infty} |d(k)|^{\frac{1}{k}} \le \limsup_{k\to\infty} |k|^{\frac{1}{k}}$ implying that the limit is 1.

b. *Theorem.*

$$\sum_{n=1}^{\infty} d(n)z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$$

Proof.[5]

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{1-z^n}}_{\text{geo. series w/ratio } z^n} z^n = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} z^{kn} z^n = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} z^{n(k+1)} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} z^{nk}$$

But then for every $m \in \mathbb{N}$, z^m appears in the double sum once for every (n,k) pair such that $nk = m$, so we indeed have $\sum_{n=1}^{\infty} d(n)z^n$.

c. *Proposition.* For $|z| < 1$,

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

Proof. We can expand a power series for the logarithm at $z = 1$ (it's awkward to do at $z = 0$ because of the branch cut):[6]

$$\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k}$$

Then multiplying by -1 shifts the negative signs onto the odd terms:

$$-\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k+2} \frac{z^k}{k} = \sum_{k=1}^{\infty} (-1)^k \frac{z^k}{k}$$

Then substituting $-z$ for z makes them go away:

$$-\log(1-z) = \sum_{k=1}^{\infty} (-1)^k \frac{(-z)^k}{k} = \sum_{k=1}^{\infty} \cancel{(-1)^k(-1)^k} \frac{z^k}{k} = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

which is *quod erat demonstrandum.*

d. *Proposition.* For $r \in (0,1)$,

$$|F(r)| \ge \frac{1}{1-r} \log\left(\frac{1}{1-r}\right)$$

Proof.[7] It turns out that we can bound each term in $F(r) = \sum_{n=1}^{\infty} \frac{r^n}{1-r^n}$ as follows.

A standard result for finite geometric series states that:

$$\frac{1-r^n}{1-r} = \sum_{k=0}^{n-1} r^k$$

For $r \in (0,1)$, $\sum_{k=0}^{n-1} r^k < n$, so we have $1 - r^n < n(1-r)$ and (taking the reciprocal) $\frac{1}{1-r^n} > \frac{1}{n(1-r)}$ and (multiplying by r^n) $\frac{r^n}{1-r^n} > \frac{r^n}{n(1-r)}$. But the left hand side is an individual term of $F(r)$, what "luck"! Then

$$\sum_{n=1}^{\infty} \frac{r^n}{1-r^n} > \sum_{n=1}^{\infty} \frac{r^n}{n(1-r)} = \frac{1}{1-r}\sum_{n=1}^{\infty} \frac{r^n}{n} = \frac{1}{1-r} \cdot -\log(1-r) = \frac{1}{1-r}\log(\frac{1}{1-r})$$

[4] Thanks to Ophir Horovitz for this insight.

[5] Thanks to Claude Sonnet 3.7 for guidance.

[6] This is pointed out in Tristan Needham's *Visual Complex Analysis*, p. 100–101,113.

[7] Thanks to OpenAI o3-mini-high for detailed guidance.