Assignment #2 (revised)

Zack M. Davis

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Abstract

Homework exercises for Prof. Lai's "Theory of Functions of a Complex Variable." Revisions in blue.

1 (Ch 1. #9). Our Cartesian Cauchy–Reimann equations are $\begin{cases} \frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \end{cases}$. We want to show that the polar form will be $\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \end{cases}$. We know that polar coördinates amount to $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. Differentiating with respect to r and θ, we have $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, and $\frac{\partial y}{\partial \theta} = r\cos\theta$.

By the chain rule, $\frac{\partial u(x,y)}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$.

Meanwhile, $\frac{\partial u(x,y)}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta = r\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta\right)$.

And $\frac{\partial v(x,y)}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cdot\cos\theta + \frac{\partial v}{\partial y}\cdot\sin\theta$

And $\frac{\partial v(x,y)}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}\cdot r\sin\theta + \frac{\partial v}{\partial y}\cdot r\cos\theta = r\left(\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right)$.

Indeed, we see that $\frac{1}{r}\cdot\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial u}{\partial r}$ ✓

And $\frac{1}{r}\cdot\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta = -\frac{\partial v}{\partial x}\cdot\cos\theta - \frac{\partial v}{\partial y}\cdot\sin\theta = -\frac{\partial v}{\partial r}$ ✓

Then we want to show that $f := \underbrace{\log r}_{u} + \underbrace{i\theta}_{iv}$ is holomorphic with $r > 0$ and $\theta \in (-\pi, \pi)$.

We have $\frac{\partial u}{\partial r} = \frac{1}{r}$ and $\frac{\partial v}{\partial \theta} = 1 \to \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} = \frac{1}{r}$ ✓.

And $\frac{\partial u}{\partial \theta} = 0$ while $-\frac{\partial v}{\partial r} = 0$ ✓ Thus f satisfies the (polar) Cauchy–Reimann equations. Furthermore, u (the reciprocal) is continuously differentiable for $r > 0$, and v (linear) is continuously differentiable, so we conclude (by Theorem 2.4) that the function is holomorphic on the given domain.

2 (Ch. 1 #10). We know that $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$. We compute that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{1}{2}(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y})\frac{1}{2}(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}) = (\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y})(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}) = \frac{\partial}{\partial x}^2 + \cancel{\frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y}} - \frac{1}{i^2}\frac{\partial}{\partial y}^2 = \frac{\partial}{\partial x}^2 + \frac{\partial}{\partial y}^2$$

as expected.

3 (Ch. 1 #11). *Theorem.* If $f = u(x,y) + iv(x,y)$ is holomorphic on open Ω, then u and v are harmonic.

Proof. The Cauchy–Reimann equations $\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}$ and $\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$ hold where f is holomorphic. Take $\frac{\partial}{\partial x}$ of the first equation and $\frac{\partial}{\partial y}$ of the second to get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$. The mixed partials are equal by Clairaut's theorem, so we have $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Similarly, take $\frac{\partial}{\partial y}$ of the first *&c.* to get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial^2 y}$ and $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial^2 x}$ implying $\frac{\partial^2 v}{\partial^2 y} = -\frac{\partial^2 v}{\partial^2 x} \Rightarrow \frac{\partial^2 v}{\partial^2 y} + \frac{\partial^2 v}{\partial^2 x} = 0$. (Thanks to Claude Sonnet 3.5 for a hint to get me unstuck.)

4 (Ch. 1 #12). *Proposition.* $f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy–Reimann equations at the origin, but is not holomorphic.

Proof. For $f(x+iy) = \underbrace{(|x||y|)^{1/2}}_{u} + \underbrace{i0}_{iv}$, we compute $\frac{\partial u}{\partial x} = \frac{1}{2}|y|\,(|x||y|)^{-1/2}$, and then $\frac{\partial u}{\partial x}$ is 0 at $x = y = 0$, and $\frac{\partial v}{\partial y} = 0$ regardless, and similarly $\frac{\partial u}{\partial y} = \frac{1}{2}|x|\,(|x||y|)^{-1/2}$ will be 0 and $\frac{\partial v}{\partial x} = 0$, so the C–R equations are satisfied by all the partials being zero. (At first I was worried that the partial derivatives didn't even exist, analogously to how $y = |x|$ isn't differentiable at the origin, but at office hours on 12 February, Prof. Lai pointed out that both coördinates being zero averts this.) Because $\{0\}$ is not open, the C–R equations being satisfied doesn't imply holomorphicity. Indeed, the function is not holomorphic because the value of the directional limit taken along *e.g.* $y = x$ depends on which direction we approach.

$$\lim_{r\to 0} \frac{f(0+r\exp(\frac{\pi}{4}i))-f(0)}{r\exp(\frac{\pi}{4}i)} = \lim_{r\to 0} \frac{f(r\cos(\frac{\pi}{4})+r\sin(\frac{\pi}{4}))-f(0)}{r\exp(\frac{\pi}{4}i)} = \lim_{r\to 0} \frac{\sqrt{|r\cos(\frac{\pi}{4})||r\sin(\frac{\pi}{4})|}-\sqrt{|0||0|}}{r\exp(\frac{\pi}{4}i)} = \lim_{r\to 0} \frac{\frac{1}{2}r}{(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)r} = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}i}{4} \text{ but } \lim_{r\to 0} \frac{f(0+r\exp(\frac{5\pi}{4}i))-f(0)}{r\exp(\frac{5\pi}{4}i)} = \lim_{r\to 0} \frac{\frac{1}{2}r}{(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)r} = -\frac{\sqrt{2}}{4} + \frac{\sqrt{2}i}{4} \neq \frac{\sqrt{2}}{4} - \frac{\sqrt{2}i}{4}.$$

5 (Ch. 1 #13). **a**. *Proposition.* If f is holomorphic and $\mathrm{Re}(f)$ is constant, then f is constant.

Proof. If $f = u(x,y) + iv(x,y)$ and u is constant, then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. The Cauchy–Reimann equations imply that $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = -0 = 0$. So v is also constant.

b. *Proposition.* If f is holomorphic and $\mathrm{Im}(f)$ is constant, then f is constant.

Proof. Just as above: v being constant implies $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, *&c.*

c. *Proposition.* If f is holomorphic and $|f|$ is constant, then f is constant.

Proof. $|f| = \sqrt{u^2+v^2} = c \Rightarrow |f|^2 = u^2 + v^2 = c$. Differentiating by x, we get $2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} = -v\frac{\partial v}{\partial x}$ and similarly differentiating by y, we get $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \Rightarrow u\frac{\partial u}{\partial y} = -v\frac{\partial v}{\partial y}$.

Switching to subscript notation to reduce Leibnizian clutter and cognitive overload, that's $uu_x = -vv_x$ and $uu_y = -vv_y$.

Invoking the C–R equations on the latter, we get $u(-v_x) = -vu_x$, which is $uv_x = vu_x$. Supposing that $u \neq 0$, that's $v_x = \frac{v}{u}u_x$.

Substituting into the previous, we have $uu_x = -v\frac{v}{u}u_x \Rightarrow uu_x + v\frac{v}{u}u_x = 0 \Rightarrow \frac{u^2+v^2}{u}u_x = 0$. We've supposed that $u \neq 0$, so if $c \neq 0$, we have $u_x = 0$. But then our earlier $vu_x = uv_x$, in conjunction with $u \neq 0$, implies $v_x = 0$. But one of the C–R equations says $u_x = 0$ implies $v_y = 0$ and the other says that $v_x = 0$ implies $u_y = 0$. If all the partials are zero, the function is constant. (Thanks to Prof. Lai at office hours on 12 February for a hint and, with deep embarrassment, Claude Sonnet 3.5 and ChatGPT o3-mini-high for extensive further tutoring.)

6. *Theorem.* For a holomorphic function $f = u(x,y) + iv(x,y)$ with nonvanishing derivative, the tangent lines to the level curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are perpendicular.

Proof. Following the hint, we have $\frac{dy}{dx} = \frac{-u_x}{u_y}$ (for the c_1 level set) and $\frac{dy}{dx} = \frac{-v_x}{v_y}$ (for the other). By the C–R equations, the latter is $\frac{u_y}{u_x}$. The product of the two slopes $\frac{-u_x}{u_y} \cdot \frac{u_y}{u_x} = -1$ implies that they are perpendicular.

7 (Ch. 1 #19). **a**. *Proposition.* $\sum nz^n$ does not converge at any point of the unit circle.

Proof. Use polar coördinates: $\sum n(r\exp(\theta i))^n = \sum nr^n\exp(n\theta i)$. On the unit cricle, $r = 1$, so $r^n = 1$ for any $n \in \mathbb{N}$, so the terms don't go to zero, so the series diverges by the divergence test.

b. *Proposition.* $\sum \frac{z^n}{n^2}$ converges at every point of the unit circle.

Proof. On the $|z| = 1$ unit circle, $\sum_{k=1}^{\infty} \left|\frac{z^n}{n^2}\right| = \sum_{k=1}^{\infty} \frac{1}{n^2}$, which converges, and absolutely convergent series are convergent.

c. *Proposition.* $\sum \frac{z^n}{n}$ converges at every point of the unit circle except $z = 1$.

Proof. If $z = 1$, we have the harmonic series $\sum \frac{1}{n}$, which is known to diverge.

If $z \neq 1$, following a hint from office hours on 12 February, we imitate the proof of Dirichlet's test in Thomas W. Wade's *An Introduction to Analysis*. We rely on the following

Lemma. $\sum_k \exp(\theta ki)$ is bounded for $\theta \neq 0$. (This should be plausible. If, say, $\theta = \frac{\pi}{4}$, then terms of the form, say, $\frac{\pi}{2} + 2\pi n$ for $n \in \mathbb{N}$ will get "cancelled out" by terms of the form $\frac{3\pi}{2} + 2\pi n$, such that the sum stays bounded even if it never converges. No matter what θ is, if we keep taking multiples $k\theta$, the average direction is zero.)

Proof (lemma). Consider the partial sum $S_n = \sum_{k=1}^{n}\exp(k\theta i)$. We multiply both sides by $(\exp(\theta i) - 1)$ to engineer a "telescoping" effect: $(\exp(\theta i) - 1)S_n = (\exp(\theta i) - 1)\sum_{k=1}^{n}\exp(k\theta i) = \sum_{k=1}^{n}\exp((k+1)\theta i) - \exp(k\theta i) = \exp((n+1)\theta i) - \exp(\theta i)$, so $S_n = \frac{\exp((n+1)\theta i)-\exp(\theta i)}{\exp(\theta i)-1}$. Both terms in the numerator are bounded by 1, so the magnitude of their difference can't be greater than 2, and the denominator only depends on θ (not n), so S_n is bounded. (Thanks to Claude Sonnet 3.5 for tutoring.)

Proof (theorem). Let M be a bound on $\sum_k |\exp(\theta ki)|$.

Fix ε. Choose $N \in \mathbb{N}$ such that $k \geq N$ implies that $\frac{1}{k} < \frac{\varepsilon}{M}$. Then for all $n, m \geq N$, we can sum by parts:

$$\left|\sum_{k=m}^{n}\frac{\exp(\theta k i)}{k}\right| \leq \left|\sum_{k=m}^{n}\exp(\theta k i)\right|\left|\frac{1}{n}\right| + \sum_{k=m}^{n-1}\left|\sum_{j=k}^{m}\exp(\theta j i)\right|\left(\frac{1}{k}-\frac{1}{k+1}\right) \leq M\frac{1}{n} + M\left(\frac{1}{m}-\frac{1}{n}\right) = M\frac{1}{m} < \varepsilon$$