Assignment #10

Zack M. Davis

17 May 2025 (late and unfinished)

Abstract

Homework submission for Prof. Schuster's "Measure and Integration" class.

§4B

1. *Proposition.* If $f \in \mathcal{L}^1$, then $\lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| = 0$.

Proof.[1]

$$\lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t,b+t]}|$$

$$= \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t,b+t]} + f(b) - f(b)| = \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b) + f(b) - f_{[b-t,b+t]}|$$

$$\leq \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| + |f(b) - f_{[b-t,b+t]}| = \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| + \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f(b) - \frac{1}{2t} \int_{[b-t,b+t]} f|$$

The first term is zero by Lebesgue's differentiation theorem (first version). The second term is also zero because $\lim_{t\downarrow 0} \frac{1}{2t} \int_{[b-t,b+t]} f = f(b)$.

3. *Proposition.* If $f^2 \in \mathcal{L}^1$, then $\lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = 0$

Proof.

$$\lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)|^2 = \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} (f - f(b))^2 = \lim_{t\downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} f^2 - 2ff(b) + f(b)^2$$

... (**UNFINISHED**, with regrets)

4. (**NOT DONE**, with regrets)

5. (**NOT DONE**, with regrets)

6. *Proposition.* If $h \in \mathcal{L}^1$ and for all s, $\int_{-\infty}^{s} h = 0$, then for almost every s, $h(s) = 0$.

Proof. Let $g(s) := \int_{-\infty}^{s} h$. By the second version of Lebesgue's differentiation theorem, $g'(s) = h(s)$ for almost every s. But $g(s) = 0$, so $g'(s) = 0$ too, so $h(s)$ is 0 for almost every s.

§6C

3. *Proposition.* $f((a_j)_{j=1}^{n}) := \sum_j |a_j|^{1/2}$ is not a norm.

Proof. We know that homogeneity will fail; we must confirm this.

The first obvious thing to try is an arbitrary example: consider $\vec{\boldsymbol{a}} := \begin{bmatrix} 4 \\ 9 \end{bmatrix}$. Then $f(\vec{\boldsymbol{a}}) = 2 + 3 = 5$, but $f(2\vec{\boldsymbol{a}}) = \sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$, and $f(2) = \sqrt{2}$ ... interestingly, homogeneity *would* be satisfied here if the axiom were the intuitively appealing $\|\alpha g\| = \|\alpha\|\|g\|$ (treating a scalar as a 1-vector): $f(2\vec{\boldsymbol{a}}) = 5\sqrt{2} = f(2)f(\vec{\boldsymbol{a}})$. But actually, the homogeneity axiom is $\|\alpha g\| = |\alpha|\|g\|$ (with an absolute value on the scalar on the right):

[1] Thanks to Prof. Schuster for discussion at office hours on 13 May.

$f(2\vec{\boldsymbol{a}}) = 5\sqrt{2} \neq 2 \cdot 5 = 2f(\vec{\boldsymbol{a}})$.

Proposition. $f((a_j)_{j=1}^n) := \left(\sum_j |a_j|^{1/2}\right)^2$ is not a norm.

Proof. The triangle inequality will fail; we confirm this with a counterexample: $f(2,2,2) = \left(3\sqrt{2}\right)^2 = 18$ but $f(0,1,1) + f(1,0,1) + f(1,1,0) = 3 \cdot (2 \cdot 1)^2 = 12$.

6. *Proposition.* Bounded functions from X to $\mathbb{F}$ with $\|f\| := \sup_X f$ is a Banach space.

Proof. (Positive-definiteness.) If $f = 0$, then $\|f\| = \sup_X |f| = 0$. But also, if $\|f\| = \sup_X |f| = 0$, then we must have $f = 0$, because if not, then the value $|f|$ takes on any point $x \in X$ where $f(x) \neq 0$ would rule out 0 as an upper bound of $|f|$. ✓

(Homogeneity.) $\|\alpha f\| = \sup_X |\alpha f| = |\alpha| \sup_X |f|$. ✓

(Triangle inequality.) $\|f + g\| = \sup_X |f + g|$. Then we can leverage the triangle inequality in $\mathbb{F}$: for all x, $|f(x)+g(x)| \leq |f(x)|+|g(x)|$, so $\sup_X |f(x)+g(x)| \leq \sup_X |f(x)|+|g(x)| \leq \sup_X |f(x)|+\sup_X |g(x)| = \|f\|+\|g\|$. ✓

(Completeness.) A Cauchy sequence $\{f_j\}$ is such that for all ε, there exists an N, such that if $m, n \geq N$, $\|f_m - f_n\| < \varepsilon$. But the uniform norm implies that if $\|f_m - f_n\| < \varepsilon$, then $\sup_x |f_m(x) - f_n(x)| < \varepsilon$ ... (**UNFINISHED**, with regrets)

7. *Proposition.* ℓ_1 with the norm $\|(a_k)_{k=1}^\infty\| = \sup_{k \in \mathbb{N}^+} |a_k|$ is not a Banach space.

Proof. Let the sequence A_j consist of j 1s followed by infinitely many 0s. Then $\{A_j\}_{j=1}^\infty$ converges in ℓ_1 with the sup norm, because for any j, $\|A_j\| = 1$, so for any m and n $\|A_m - A_n\| = 0$. But $\lim_{j\to\infty} A_j$ is the all-ones sequence, which is not in ℓ_1.