Assignment #9

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Abstract

Homework submission for Prof. Schuster's "Measure and Integration" class.

§4B

1. *Proposition.* For all positive c and p

$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c^p} \int |h|^p \, d\mu$$

Proof.

$$\mu(\{x \in X : |h(x)| \geq c\}) = \int \mathbf{1}_{\{x \in X : |h(x)| \geq c\}} \, d\mu = \underbrace{\frac{c^p}{c^p}}_{\text{fancy one}} \int \mathbf{1}_{\{x \in X : |h(x)| \geq c\}} \, d\mu = \frac{1}{c^p} \int \mathbf{1}_{\{x \in X : |h(x)| \geq c\}} c^p \, d\mu$$

But on the domain where $|h(x)| \geq c$, then $|h|^p \geq c^p$.

$$\leq \frac{1}{c^p} \int \mathbf{1}_{\{x \in X : |h(x)| \geq c\}} |h|^p \, d\mu \leq \frac{1}{c^p} \int |h|^p \, d\mu$$

which is *quod erat demonstrandum.*

2. *Theorem* (Chebyshev's inequality). For a measure space (X, Σ, μ) with $\mu(X) = 1$ and $h \in \mathcal{L}^1$, then for all $c > 0$,

$$\mu\left(\left\{x \in X : \left|h(x) - \int h \, d\mu\right| \geq c\right\}\right) \leq \frac{1}{c^2}\left(\int h^2 \, d\mu - \left(\int h \, d\mu\right)^2\right)$$

Proof attempt.

$$\mu\left(\left\{x \in X : \left|h(x) - \int h \, d\mu\right| \geq c\right\}\right) = \int \mathbf{1}_{\{x \in X : |h(x) - \int h \, d\mu| \geq c\}} d\mu = \frac{c^2}{c^2} \int \mathbf{1}_{\{x \in X : |h(x) - \int h \, d\mu| \geq c\}} d\mu$$

But then if $\left|h(x) - \int h \, d\mu\right| \geq c$, then $\left|h(x) - \int h \, d\mu\right|^2 \geq c^2$

Also, $\left|h(x) - \int h \, d\mu\right|^2 = \left(h(x) - \int h \, d\mu\right)^2 = h^2(x) - 2h(x)\left(\int h \, d\mu\right) + \left(\int h \, d\mu\right)^2$. (**UNFINISHED**, with regrets)

3. *Proposition.* For a measure space (X, Σ, μ) and $h \in \mathcal{L}^1$and $\|h\|_1 > 0$, there is at most one $c > 0$ such that

$$\mu(\{x \in X : |h(x)| \geq c\}) = \frac{1}{c}\|h\|_1$$

Proof attempt. Suppose that $\mu(\{x \in X : |h(x)| \geq c\}) = \frac{1}{c}\|h\|_1$ and $\mu(\{x \in X : |h(x)| \geq d\}) = \frac{1}{d}\|h\|_1$. Then $\frac{1}{c}\int \mathbf{1}_{\{x \in X : |h(x)| \geq c\}} c \, d\mu = \frac{1}{d}\int \mathbf{1}_{\{x \in X : |h(x)| \geq d\}} d \, d\mu$ (**UNFINISHED**, with regrets)

4. *Proposition.* The constant 3 in the Vitali covering lemma is minimal.

Proof. For some ε_1, consider $I_1 := (0, 1 + \varepsilon_1)$, $I_2 := (1, 2)$, and $I_3 := (2 - \varepsilon_1, 3)$. Notice that $(3 - \varepsilon) * I_2 = (0 + \frac{\varepsilon}{2}, 3 - \frac{\varepsilon}{2})$ does not suffice to cover $\bigcup_{k=1}^{3} I_k = (0, 3)$. (**UNFINISHED**, with regrets: uh, but why doesn't $I_1 \cup I_3$

work? There's something subtle here with the two epsilons ... notice that the proof is greedy, but my candidate here is non-greedy?)

9. *Proposition.* For Lebesgue measurable h and $c \in \mathbb{R}$, $\{b \in \mathbb{R} : h^*(b) > c\}$ is open.

Proof. Recall that the Hardy–Littlewood maximal function is defined as

$$h^*(b) := \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h|$$

Fix ε. By Axler's 3.28 ("integrals on small sets are small"), there exists a δ such that for all B with $\mu(B) < \delta$, then $\int_{b-t}^{b+t} |h|$

Note that $\{b \in \mathbb{R} : h^*(b) > c\}$ is the inverse image $h^{*-1}((c, \infty))$.

10. *Proposition.* If $h : \mathbb{R} \to [0, \infty)$ is nondecreasing, then so is h^*.

Proof. If h is nondecreasing, that means that if $a < b$, then $h(a) \leq h(b)$. Suppose $a < b$.

For any given t, $\frac{1}{2t} \int_{a-t}^{a+t} |h| \leq \frac{1}{2t} \int_{b-t}^{b+t} |h|$, because $\int_{a-t}^{b-t} |h| \leq$ (**UNFINISHED**, with regrets)

12. *Proposition.* If $h \in \mathcal{L}^1$, then$|\{b \in \mathbb{R} : h^*(b) = \infty\}| = 0$.

Proof. For any c, $\{b \in \mathbb{R} : h^*(b) = \infty\} \subseteq \{b \in \mathbb{R} : h^*(b) \geq c\}$. By the monotonicity of outer measure and the Hardy–Littlewood inequality, $|\{b \in \mathbb{R} : h^*(b) = \infty\}| \leq |\{b \in \mathbb{R} : h^*(b) \geq c\}| \leq \frac{3}{c} \|h\|_1$.

Fix ε. Let $C := \frac{3\|h\|_1}{\varepsilon}$. Then if $c > C$, $\frac{3}{c} \|h\|_1 < \frac{3}{\frac{3\|h\|_1}{\varepsilon}} \|h\|_1 = \cancel{\frac{\varepsilon}{3\|h\|_1} 3 \|h\|_1} = \varepsilon$.