# Assignment #7

17 March 2025

**Abstract**

Homework exercises for Prof. Dusty Ross's "Modern Algebra I".

**1**. *Proposition.* $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group.

*Proof.* Suppose it were. Then it would be generated by a single element. Suppose that element took the form $g := (a, b)$ for $a, b \in \mathbb{Z}$. But $(-a, b) \notin \langle (a, b) \rangle$: you need to take $-1g$ to get $-a$ in the first coördinate, but $b$ is not the second coördinate of $-1g$. Contradiction!

**2**. *Proposition.* $\langle (5, 10) \rangle$ and $\langle (10, 10) \rangle$ are subgroups of $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/40\mathbb{Z}$ with order 12.

*Discussion.* I see what this exercise is trying to do: any subgroup of order 12 is going to have to involve "components" from both $\mathbb{Z}/30\mathbb{Z}$ and $\mathbb{Z}/40\mathbb{Z}$ (because 40 doesn't have any factors of 3 and 30 doesn't have enough factors of 2).

*Computation.* We compute $\langle (5, 10) \rangle$
$= \{(5, 10), (10, 20), (15, 30), (20, 0), (25, 10), (0, 20), (5, 30), (10, 0), (15, 10), (20, 20), (25, 30), (0, 0)\}$.
And likewise $\langle (10, 10) \rangle$
$= \{(10, 10), (20, 20), (0, 30), (10, 0), (20, 10), (0, 20), (10, 30), (20, 0), (0, 10), (10, 20), (20, 30), (0, 0)\}$.

**3**. **a**. *Proposition.* $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is isomorphic to $\mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

*Proof.* By Corollary 2 of Gallian's Theorem 8.2, they're both isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

**b**. *Proposition.* $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

*Proof.* The latter is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. The order-four cyclic groups can't be further factorized.

**4**. *Proposition.* $\langle (1\ 2\ 3\ 4) \rangle$ is not a normal subgroup of $S_4$.

*Proof.* We compute the elements of the subgroup $(1\ 2\ 3\ 4)(1\ 2\ 3\ 4) = (4\ 2)(1\ 3) = (1\ 3)(2\ 4)$; $(1\ 3)(2\ 4)(1\ 2\ 3\ 4) = (1\ 4\ 3\ 2)$; $(1\ 4\ 3\ 2)(1\ 2\ 3\ 4) = (1)(2)(3)(4) = 1$.

This allows us to compute the cosets, and we find that they are not equal. This was found via a Python program (modified from a program written for assignment #5, new code reproduced below, shared code omitted):

```
def generate_subgroup(element):
    working = element
    h = set()
    for i in range(10):
        new = working * element
        h.add(new)
        working = new
    return h

if __name__ == "__main__":
    h = generate_subgroup(Permutation([2, 3, 4, 1]))
    s_4 = [Permutation(p) for p in permutations((1, 2, 3, 4))]
    left_coset_map = defaultdict(list)
    right_coset_map = defaultdict(list)
    for element in s_4:
```

```
            left_coset = tuple(sorted(element * h_j for h_j in h))
            left_coset_map[left_coset].append(element)
            right_coset = tuple(sorted(h_j * element for h_j in h))
            right_coset_map[right_coset].append(element)
        for i, coset_map in enumerate([left_coset_map, right_coset_map]):
            for coset, elements in coset_map.items():
                print(
                    " = ".join(
                        [
                            "{" + ", ".join(repr(coset_member) for coset_member in coset) + "}",
                            *[repr(element) + "H" if i == 0 else "H" +
                              repr(element) for element in elements],
                        ]
                    )
                )
```

The program reveals that

```
zmd@system76-pc:~/Documents/School/Algebra$ python3 symmetric_cosets.py
{1, (1 2 3 4), (1 3)(2 4), (1 4 3 2)} = 1H = (1 2 3 4)H = (1 3)(2 4)H = (1 4 3 2)H
{(3 4), (1 2 3), (1 3 2 4), (1 4 2)} = (3 4)H = (1 2 3)H = (1 3 2 4)H = (1 4 2)H
{(2 3), (1 2 4), (1 3 4 2), (1 4 3)} = (2 3)H = (1 2 4)H = (1 3 4 2)H = (1 4 3)H
{(2 3 4), (1 2 4 3), (1 3 2), (1 4)} = (2 3 4)H = (1 2 4 3)H = (1 3 2)H = (1 4)H
{(2 4 3), (1 2), (1 3 4), (1 4 2 3)} = (2 4 3)H = (1 2)H = (1 3 4)H = (1 4 2 3)H
{(2 4), (1 2)(3 4), (1 3), (1 4)(2 3)} = (2 4)H = (1 2)(3 4)H = (1 3)H = (1 4)(2 3)H
{1, (1 2 3 4), (1 3)(2 4), (1 4 3 2)} = H1 = H(1 2 3 4) = H(1 3)(2 4) = H(1 4 3 2)
{(3 4), (1 2 4), (1 3 2), (1 4 2 3)} = H(3 4) = H(1 2 4) = H(1 3 2) = H(1 4 2 3)
{(2 3), (1 2 4 3), (1 3 4), (1 4 2)} = H(2 3) = H(1 2 4 3) = H(1 3 4) = H(1 4 2)
{(2 3 4), (1 2), (1 3 2 4), (1 4 3)} = H(2 3 4) = H(1 2) = H(1 3 2 4) = H(1 4 3)
{(2 4 3), (1 2 3), (1 3 4 2), (1 4)} = H(2 4 3) = H(1 2 3) = H(1 3 4 2) = H(1 4)
{(2 4), (1 2)(3 4), (1 3), (1 4)(2 3)} = H(2 4) = H(1 2)(3 4) = H(1 3) = H(1 4)(2 3)
```

Whereupon we observe that, *e.g.*, $(1\ 2)H \neq H(1\ 2)$.

**5**. *Proposition*. Suppose $H \leq G$ and $|G|/|H| = 2$. Then $H \trianglelefteq G$.

*Proof.* There are two left and two right cosets of $H$, because Lagrange ($|G|/|H| = |G : H|$). On the left, one contains the identity, and the other contains the elements $G \backslash H$. Ditto on the right. So the left and right cosets are the same.