Analogues of π in the L^p-spaces over $\mathbb{R}^n$

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It is a fact so well-known so as to be scarcely worth mentioning (even amongst those who have no interest in mathematics) that the ratio of the circumference of a circle to its diameter is a constant (denoted π) approximately equal to 3.14159265. What is less well-known are the conditions on which this theorem depends. Specifically, while we usually equip $\mathbb{R}^2$ with the Euclidean metric for the perfectly good reason that the geometry of the real world is extremely Euclidean, there are other ways to define "distance" in the plane that satisfy the metric space axioms, and these alternative metrics bring with them alternative "circles" (scare quotes omitted hereafter) whose cirumference-to-diameter ratios can be different. In this document, I present a formula for the circumference-to-diameter ratios of circles under the metric induced by the p-norm. I will denote such ratios with the symbol π_p.

We begin by reviewing the definition of the p-norm.

Definition. The p-norm on $\mathbb{R}^n$ for $1 \leq p \leq \infty$ is defined as

$$||\vec{\mathbf{x}}|| = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

The normed vector space using the p-norm is sometimes called the *L^p-space over* $\mathbb{R}^n$, and I will also use this term for the corresponding metric space.

Note that for $p := 2$, the p-norm is exactly the Euclidean norm, so $\pi_2 = \pi$.

There is a sense in which the L^p-spaces for $p :\neq 2$ are distressingly *ugly* in that they do not enjoy the rotational invariance possessed by Euclidean space, but we press on in our chosen task regardless. To find π_p, we will want a parametrization of a circle in this space. In fact, for convenience, we can just consider the restriction of the unit circle to the first quadrant of the plane, as in the following

Proposition. The restriction to the first quadrant of the unit circle in the L^p-space over $\mathbb{R}^n$ is parametrized by

$$\begin{cases} x(\theta) = \cos^{2/p} \theta \\ y(\theta) = \sin^{2/p} \theta \end{cases}$$

Proof. We see that $(x(\theta)^p + y(\theta)^p)^{1/p} = \left(\cos^2\theta + \sin^2\theta\right)^{1/p} = 1$, but this is sufficient.

At this juncture, we now have a satisfactory notion of *linear* distance between points in L^p-space, but we still need to understand how the length of an *curve* is to be calculated. In Euclidean space, arc length of a parametrized curve $\vec{\mathbf{r}}(t) = [x_j(t)]_{j=1}^n$ from $t := a$ to $t := b$ is given by the limit of a Riemann sum of segments approximating the curve, computed as the integral $\int_a^b \sqrt{\sum_j \left(\frac{dx_j}{dt}\right)^2}\, dt$. The generalization to L^p-space is very intuitive, and is stated (albeit without proof) in the following

Proposition. The length of a parametrized curve $\vec{\mathbf{r}}(t)$ in the L^p-space over $\mathbb{R}^n$ from $t := a$ to $t := b$ is given by

$$\int_a^b \left(\sum_{j=1}^n \left|\frac{dx_j}{dt}\right|^p\right)^{1/p} dt.$$

We take the absolute value of $\frac{dx_j}{dt}$ because (as in the definition of the p-norm itself) in computing the length of a segment, we want to know only the magnitude of the change in each coordinate, without regard to direction. (The absolute value sign is almost never written in the Eudclidean case because it would be redundant.)

With the above preliminaries established, the theorem of interest is now easily demonstrated.

Theorem. The ratio π_p of the circumference to the diameter of a circle under the p-norm is given by

$$\frac{4}{p}\int_0^{\pi/2} \left(\cos^{2-p}\theta\sin^p\theta + \sin^{2-p}\theta\cos^p\theta\right)^{1/p} d\theta$$

Proof. The ratio π_p will be given by twice the arc length of the unit circle in the first quadrant. This is

$$\pi_p = 2\int_0^{\pi/2} \left(\left|\frac{dx}{d\theta}\right|^p + \left|\frac{dy}{d\theta}\right|^p\right)^{1/p} d\theta.$$

But the derivatives $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ are easily computed as

$$\frac{dx}{d\theta} = -\frac{2}{p}\cos^{\frac{2-p}{p}}\theta\sin\theta$$

$$\frac{dy}{d\theta} = \frac{2}{p}\sin^{\frac{2-p}{p}}\theta\cos\theta.$$

which (after some elementary algebraic manipulations) gives us

$$\pi_p = \frac{4}{p}\int_0^{\pi/2} \left(\cos^{2-p}\theta\sin^p\theta + \sin^{2-p}\theta\cos^p\theta\right)^{1/p} d\theta,$$

which is QUOD ERAT DEMONSTRANDUM.

Numerical values of π_p for various p between 1 and 2 were approximated by computer and are presented in the following table:

p	π_p
1	4
1.1	3.75708436728
1.2	3.57268318384
1.3	3.43447027313
1.4	3.33277721949
1.5	3.25976799306
1.6	3.20906831638
1.7	3.17554763089
1.8	3.15513471638
1.9	3.14464057769
2	3.14159265359